MULTIPLE FUZZY HYPOTHESES TESTING

Bülent Baygün

Schlumberger-Doll Research
Old Quarry Rd., Ridgefield, CT
e-mail: baygun@ridgefield.sdr.slb.com

ABSTRACT

We present a methodology to test multiple hypotheses on the distribution of a random variable when the hypotheses are parameterized by fuzzy variables. The proposed approach has a Bayesian flavor in the sense that the objective is to minimize a fuzzy average decision error probability by a proper choice of decision regions. We use a scalar index, called the total distance criterion (TDC) ranking index, in order to rank the fuzzy average decision error probabilities of different decision rules. We derive the optimal decision rule which minimizes the TDC index of the fuzzy average decision error probability. As an example we apply the general approach proposed here to the classification of the fuzzy mean of a Gaussian random variable.

1. OVERVIEW

In many parameteric statistical decision problems, one has to make a decision in the presence of some uncertainties about the parameters in the statistical model of the observed random variables. Examples include estimation of signal parameters in additive noise of unknown power and detection of a signal of unknown amplitude. Many powerful statistical methods exist that deal with such uncertainties including invariance techniques, min-max methods and probabilistic modeling of the unknown nuisance parameters [1]. The conventional methods assume that the unknown parameters belong to parameter classes with sharp boundaries. On the other hand, there are many examples where the very definition of a parameter class calls for graded boundaries. An example from geology is the definition of lithofacies classes based on sand and clay content: shaly sand or sandy shale are two lithofacies classes that are defined as subsets on the sand content interval of [0%, 100%]. A geologist's perception of the two lithofacies classes corresponds to graded and possibly partially overlapping subsets of [0%, 100%] rather than to disjoint subsets of the same interval with sharp cutoffs. Parameter classes that have a graded boundary can be modeled as *fuzzy sets* over an appropriate domain. When a parametric statistical decision problem involves fuzzy parameter sets of the parameter space, the conventional statistical methods cannot be used directly to derive optimal decision rules. Another approach should be adopted which recognizes the fuzziness of the parameter classes. In this paper, we present a multiple hypotheses testing framework which can be used when the hypotheses are parameterized by fuzzy variables.

Our result is a generalization of the binary fuzzy hypothesis testing result reported in [2]. We derive the optimal decision rule for M-ary fuzzy hypothesis testing and apply this rule to the classification of the fuzzy mean of a Gaussian random variable.

We start with M hypotheses parameterized by fuzzy variables. Adopting a Bayesian framework, i.e. assuming priors on the hypotheses, and using fuzzy mathematics, we show that the average error probability is a fuzzy number. The fuzzy error probability depends on the choice of the decision regions over the sample space. In order to rank the fuzzy probabilities resulting from different choices of the decision regions, we use an index on the fuzzy probabilities, called the "total distance criterion (TDC) ranking index" [2]. A decision rule which minimizes the TDC of the fuzzy average error probability is said to be "optimal". We prove that the optimal decision rule performs a comparison among the TDC indices of the fuzzy likelihood functions weighted by their prior probabilities and chooses the hypothesis for which the weighted TDC index of the fuzzy likelihood function is largest.

To illustrate the method we consider the estimation of an unknown fuzzy component $\tilde{\xi}$ in Gaussian noise, where the possible values of the fuzzy component are fuzzy numbers $\tilde{\xi}_1, \ldots, \tilde{\xi}_M$. This kind of problem arises in amplitude shift keying in communications [5], where the amplitudes are subject to drift about nominal values, in cement bond logging [4], where the amplitude of the signal indicates the quality of the cement bond but the amplitude classes do not have sharp boundaries, as well as in textural identification of rocks based on grain size [3], where the grain size has a distinct but fuzzy mean for each rock type.

2. STATEMENT AND SOLUTION OF THE MULTIPLE FUZZY HYPOTHESES TESTING PROBLEM

Throughout the text, we denote random numbers by boldface letters, such as $\mathbf{X}$, and fuzzy sets by a tilde on top of the variable symbol, such as $\tilde{\xi}$. Before we start, let us give the definitions of some fuzzy logic terms. A fuzzy set $\tilde{a}$ over $\mathcal{A}$ is defined by its membership function $\mu_{\tilde{a}}(\cdot)$ which is a mapping from $\mathcal{A}$ to the unit interval [0, 1]. The α-cut $(\tilde{a})^\alpha$ of a fuzzy set $\tilde{a}$ is defined by $(\tilde{a})^\alpha \stackrel{\text{def}}{=} \{a|\mu_{\tilde{a}}(a) > \alpha\}$. A fuzzy number is a fuzzy set $\tilde{\xi}$ over the real line $\mathcal{R}$ with the following properties: its maximum membership value is 1; its membership function is piecewise continuous; and it is a

0-7803-2431-5/95 $4.00 © 1995 IEEE

convex fuzzy set, i.e. for any $a_1, a_2 \in \mathcal{R}$ and any $\lambda \in [0,1]$, the membership function satisfies $\mu_{\tilde{a}}(\lambda a_1 + (1-\lambda)a_2) \geq \min\{\mu_{\tilde{a}}(a_1), \mu_{\tilde{a}}(a_2)\}$. The α-cut of a fuzzy number $\tilde{a}$ is specified by its left and right end points, denoted by $[\tilde{a}]_1^\alpha$ and $[\tilde{a}]_2^\alpha$, respectively. The support of a fuzzy number $\tilde{a}$ is the open interval $([\tilde{a}]_1^0, [\tilde{a}]_2^0)$.

Given a random variable $\mathbf{X}$ consider M hypotheses H_m : $\mathbf{X} \sim f_{X|m;\tilde{\xi}_m}$, $m = 1, \ldots, M$, where $f_{X|1;\tilde{\xi}_1}, \ldots, f_{X|M;\tilde{\xi}_M}$ are given probability density functions parameterized by the fuzzy numbers $\tilde{\xi}_1, \ldots, \tilde{\xi}_M$, respectively. Now for any set of fixed values $\xi_1, \ldots, \xi_M$ of the fuzzy numbers $\tilde{\xi}_1, \ldots, \tilde{\xi}_M$ and for given prior probabilities $p_1, \ldots, p_M$ on the hypotheses $H_1, \ldots, H_M$, the average error probability P_e can be written as

$$P_e = \sum_{m=1}^{M} \int_{\mathcal{R}-\mathcal{R}_m} p_m f_{X|m;\xi_m}(x)\, dx. \tag{1}$$

In (1), for each $m = 1, \ldots, M$, the set $\mathcal{R}_m$ is that subset of the real line $\mathcal{R}$ on which the hypothesis H_m is decided.

For any value $x \in \mathcal{X}$ of the random variable $\mathbf{X}$ and for any $m = 1, \ldots, M$, define the function $g_{x;m}(\cdot)$ by $g_{x;m}(\xi) \stackrel{\text{def}}{=} f_{X|m;\xi}(x)$. The function $g_{x;m}(\cdot)$ induces a fuzzy set $\tilde{g}_{x;m} \stackrel{\text{def}}{=} g_{x;m}(\tilde{\xi}_m)$ over the set of non-negative real numbers. Given the fuzzy set $\tilde{\xi}_m$ and the function $g_{x;m}(\cdot)$, the membership function of the fuzzy image set $\tilde{g}_{x;m}$ is found by applying the extension principle [6] which states that:

$$\mu_{\tilde{g}_{x;m}}(g) \stackrel{\text{def}}{=} \sup_{\xi:\ g = g_{x;m}(\xi)} \mu_{\tilde{\xi}_m}(\xi) \tag{2}$$

Now assume that the fuzzy set $\tilde{g}_{x;m}$ whose membership function is computed through the extension principle as in (2) is a fuzzy number. Then, since any linear combination of non-negative fuzzy numbers using positive coefficients results in a non-negative fuzzy number, the average error probability P_e is a fuzzy number. The fuzzy average error probability will be denoted by $\tilde{P}_e$.

In order to rank the fuzzy average error probabilities $\tilde{P}_e$ resulting from different choices of the decision regions $\mathcal{R}_m$, we use an index on the fuzzy probabilities called the "total distance criterion (TDC) ranking index" [2]. The TDC index of an arbitrary fuzzy number $\tilde{a}$ is defined by

$$T(\tilde{a}) \stackrel{\text{def}}{=} \int_0^1 \frac{1}{2}([\tilde{a}]_1^\alpha + [\tilde{a}]_2^\alpha)\, d\alpha, \tag{3}$$

where $[\tilde{a}]_1^\alpha$ and $[\tilde{a}]_2^\alpha$ are the left and right end points of the α-cut of $\tilde{a}$. A decision rule which minimizes the TDC of the probability of error is said to be "optimal". The TDC index has a number of properties previously established in [2] which make it a suitable representation of fuzzy numbers. These properties include the following: i) the TDC index of a crisp number is the crisp number itself, i.e. if $\tilde{a} = a \in \mathcal{R}$, then $T(\tilde{a}) = a$; and ii) the TDC index is linear under fuzzy addition and scalar multiplication by non-negative numbers, i.e. if $\tilde{a}_1$ and $\tilde{a}_2$ are two fuzzy numbers and c is a non-negative scalar, then $T(c\tilde{a}_1 + \tilde{a}_2) = cT(\tilde{a}_1) + T(\tilde{a}_2)$. Using these properties, the TDC index of the fuzzy probability $\tilde{P}_e$ can be written as:

$$T(\tilde{P}_e) = \sum_{m=1}^{M} \int_{\mathcal{R}-\mathcal{R}_m} p_m T(\tilde{g}_{x;m})\, dx. \tag{4}$$

The following theorem specifies the optimal non-randomized decision rule.

Theorem 1 *The TDC index $T(\tilde{P}_e)$ of fuzzy average error probability is minimized when the decision regions $\mathcal{R}_m$ are chosen as follows:*

$$\mathcal{R}_m = \mathcal{R}_m^* \stackrel{\text{def}}{=} \{x | p_m T(\tilde{g}_{x;m}) > p_j T(\tilde{g}_{x;j}), \forall j \neq m\}\ , \quad \textit{for } m = 1, \ldots, M \tag{5}$$

Proof: Denote the fuzzy average error probability for the decision regions specified in (5) by $\tilde{P}_e^*$. We will show that if $\tilde{P}_e$ is the fuzzy average error probability for any other choice of decision regions, then $T(\tilde{P}_e) - T(\tilde{P}_e^*) \geq 0$. For an arbitrary set of decision regions $\mathcal{R}_1, \ldots, \mathcal{R}_M$, the TDC index of the fuzzy average error probability can be written as:

$$T(\tilde{P}_e) = \sum_{m=1}^{M} \int_{\mathcal{R}} p_m T(\tilde{g}_{x;m})\, [1 - \phi_m(x)]\, dx\ , \tag{6}$$

where $\phi_m(x)$ is the indicator function of $\mathcal{R}_m$ for $m = 1, \ldots, M$, i.e. $\phi_m(x) = 1$ if $x \in \mathcal{R}_m$ and $\phi_m(x) = 0$ otherwise. For any $x \in \mathcal{R}$, there is only one $m \in \{1, \ldots, M\}$ for which $\phi_m(x) \neq 0$. Hence the TDC equation (6) can be equivalently written in the following form:

$$T(\tilde{P}_e) = \sum_{m=1}^{M} \int_{\mathcal{R}} p_m T(\tilde{g}_{x;m})\, [1 - \phi_m(x)] \sum_{n=1}^{M} \phi_n^*(x)\, dx\ . \tag{7}$$

Therefore,

$$T(\tilde{P}_e) - T(\tilde{P}_e^*) = \sum_{m=1}^{M} \sum_{n=1}^{M} \int_{\mathcal{R}} (p_m T(\tilde{g}_{x;m}) - p_n T(\tilde{g}_{x;n}))\, \phi_n(x) \phi_m^*(x)\, dx \tag{8}$$

Now $\phi_m^*(x) = 0$ unless $m = m_{max} \stackrel{\text{def}}{=} \arg\max_j p_j T(\tilde{g}_{x;j})$. Hence

$$T(\tilde{P}_e) - T(\tilde{P}_e^*) = \sum_{n=1}^{M} \int_{\mathcal{R}} \overbrace{(p_{m_{max}} T(\tilde{g}_{x;m_{max}}) - p_n T(\tilde{g}_{x;n}))}^{\geq 0}\, \phi_n(x) \phi_{m_{max}}^*(x)\, dx \tag{9}$$

and this proves the theorem because $\phi_n(x)\phi_{m_{max}}^*(x) \geq 0$ $\forall x \in \mathcal{R}$. (*QED*)

The optimal decision rule specified by (5) forms the main theoretical result of this paper and is an M-ary extension of the binary hypothesis test proposed in [2]. An equivalent way of stating Theorem 1 is the following: the optimal decision rule chooses that hypothesis for which the weighted TDC index of the fuzzy number $\tilde{g}_{x;m}$ is largest. When the parameters $\tilde{\xi}_m$ are non-fuzzy, $T(\tilde{g}_{x;m}) = g_{x;m}(\xi_m) = f_{X|m;\xi_m}(x)$ and the optimal decision rule (5) reduces to the maximum *a posteriori* (MAP) decision rule, as expected.

3. CLASSIFICATION OF THE FUZZY MEAN OF A GAUSSIAN RANDOM VARIABLE

Now suppose that the observed variable $\mathbf{X}$ is modeled as the sum of a fuzzy component $\tilde{\xi}$ and a zero-mean Gaussian random component $\mathbf{N}$. The objective is to classify the mean $\tilde{\xi}$ of $\mathbf{X}$ as one of M fuzzy numbers $\tilde{\xi}_1, \ldots, \tilde{\xi}_M$. Under each hypothesis H_m and for any fixed value ξ of the fuzzy number $\tilde{\xi}$, the likelihood function of $\mathbf{X}$ has the form $f_{X|m;\xi}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\xi)^2}{2\sigma^2})$. Under the m^{th} hypothesis H_m, the fuzzy number $\tilde{\xi}$ can assume values in the interval $\mathcal{X}_m \stackrel{\text{def}}{=} ([\tilde{\xi}_m]_1^0, [\tilde{\xi}_m]_2^0)$.

Proposition 1 *Let the membership function $\mu_{\tilde{\xi}_m}(\xi)$ of $\tilde{\xi}_m$ be symmetric about the mid-point $[\tilde{\xi}_m]_0 \stackrel{\text{def}}{=} ([\tilde{\xi}_m]_1^0 + [\tilde{\xi}_m]_2^0)/2$ of $\mathcal{X}_m$. Then for any value x of the random variable $\mathbf{X}$, the fuzzy set $\tilde{g}_{x;m}$ induced by the mapping $g_{x;m}(\xi) = f_{X|m;\xi}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\xi)^2}{2\sigma^2})$ is a fuzzy number.*

Proof: By virtue of the symmetry of $\mu_{\tilde{\xi}_m}(\xi)$ about $\xi = [\tilde{\xi}_m]_0$ and the symmetry of $\theta_x(\xi) \stackrel{\text{def}}{=} \frac{-1}{2\sigma^2}(x-\xi)^2$ about $\xi = x$, the fuzzy number $\tilde{g}_{x;m}$ depends only on the absolute difference $|x - [\tilde{\xi}_m]_0|$ and it suffices to consider $x \leq [\tilde{\xi}_m]_0$. We distinguish two cases: i) $x \leq [\tilde{\xi}_m]_1^0$ (x is outside and to the left of the support $\mathcal{X}_m$ of $\tilde{\xi}_m$); and ii) $[\tilde{\xi}_m]_1^0 < x \leq [\tilde{\xi}_m]_0$ (x is within the left half of $\mathcal{X}_m$). For any ξ, we have $g_{x;m}(\xi) \in (0, \frac{1}{\sqrt{2\pi\sigma^2}}]$. So, $\mu_{\tilde{g}_{x;m}}(g) = 0$ for any $g \notin (0, \frac{1}{\sqrt{2\pi\sigma^2}}]$. Suppose that $g \in (0, \frac{1}{\sqrt{2\pi\sigma^2}}]$. Then the equation $g_{x;m}(\xi) = g$ has two solutions: $\xi^{(1)}(g;x) = x - \sqrt{-2\sigma^2\ln(\sqrt{2\pi\sigma^2}g)}$ and $\xi^{(2)}(g;x) = x + \sqrt{-2\sigma^2\ln(\sqrt{2\pi\sigma^2}g)}$. Therefore the membership function of $\tilde{g}_{x;m}$, obtained via (2), is given by: $\mu_{\tilde{g}_{x;m}}(g) = \max_{i=1,2}\{\mu_{\tilde{\xi}_m}(\xi^{(i)}(g;x))\}$. First we will show that in both cases i) and ii),

$$\mu_{\tilde{\xi}_m}(\xi^{(2)}(g;x)) \geq \mu_{\tilde{\xi}_m}(\xi^{(1)}(g;x)), \tag{10}$$

such that $\mu_{\tilde{g}_{x;m}}(g) = \mu_{\tilde{\xi}_m}(\xi^{(2)}(g;x))$.

Case 1 ($x \leq [\tilde{\xi}_m]_1^0$):

$$\begin{aligned} \xi^{(1)}(g;x) &= x - \sqrt{-2\sigma^2\ln(\sqrt{2\pi\sigma^2}g)} \\ &\leq x \\ &\leq [\tilde{\xi}_m]_1^0. \end{aligned} \tag{11}$$

Therefore, $\xi^{(1)}(g;x)$ lies outside the support $\mathcal{X}_m$ of $\tilde{\xi}_m$ and $\mu_{\tilde{\xi}_m}(\xi^{(1)}(g;x)) = 0$. Since $\mu_{\tilde{\xi}_m}(\xi^{(2)}(g;x)) \geq 0$, (10) holds.

Case 2 ($[\tilde{\xi}_m]_1^0, x \leq [\tilde{\xi}_m]_0$): For $g \leq g_{x;m}([\tilde{\xi}_m]_1^0)$, again we have $\xi^{(1)}(g;x) \leq [\tilde{\xi}_m]_1^0$ and the conclusion of Case 1 is still valid. Suppose that $g_{x;m}([\tilde{\xi}_m]_1^0) < g \leq \frac{1}{\sqrt{2\pi\sigma^2}}$. Let $\delta \stackrel{\text{def}}{=} [\tilde{\xi}_m]_0 - x$ and $\Gamma(g) \stackrel{\text{def}}{=} \sqrt{-2\sigma^2\ln(\sqrt{2\pi\sigma^2}g)}$. Then $\xi^{(1)}(g;x) = x - \Gamma(g) = [\tilde{\xi}_m]_0 - (\delta + \Gamma(g))$ and $\xi^{(2)}(g;x) = x + \Gamma(g) = [\tilde{\xi}_m]_0 - (\delta - \Gamma(g))$. It can be shown that for any $\eta_1 > \eta_2 > 0$, $\mu_{\tilde{\xi}_m}([\tilde{\xi}_m]_0 \pm \eta_2) \geq \mu_{\tilde{\xi}_m}([\tilde{\xi}_m]_0 \pm \eta_1)$. Since $\delta \geq 0$ and $\Gamma(g) \geq 0$, we have $|\delta - \Gamma(g)| \leq |\delta + \Gamma(g)|$. Therefore, choosing $\eta_2 = |\delta - \Gamma(g)|$ and $\eta_1 = |\delta + \Gamma(g)|$, we obtain (10).

It follows that $\tilde{g}_{x;m}$ satisfies the three conditions for being a fuzzy number, as shown below.

1) $\mu_{\tilde{g}_{x;m}}(g)$ is a continuous function of g: $\mu_{\tilde{\xi}_m}(\xi)$ and $\xi^{(2)}(g;x)$ are continuous functions of ξ and g, respectively. Therefore, the composite function $\mu_{\tilde{g}_{x;m}}(g) = \mu_{\tilde{\xi}_m}(\xi^{(2)}(g;x))$ is a continuous function of g.

2) $\tilde{g}_{x;m}$ is a normal set: Let $g = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(\theta_x([\tilde{\xi}_m]_0))$. Then $\mu_{\tilde{g}_{x;m}}(g) = \mu_{\tilde{\xi}_m}([\tilde{\xi}_m]_0) = 1$.

3) $\tilde{g}_{x;m}$ is a convex set: Let $g_1, g_2 \in (0, \frac{1}{\sqrt{2\pi\sigma^2}}]$. Without loss of generality, assume $g_1 < g_2$. For any $\lambda \in [0,1]$,

$$\mu_{\tilde{g}_{x;m}}(\lambda g_1 + (1-\lambda)g_2) = \mu_{\tilde{\xi}_m}(\xi^{(2)}(\lambda g_1 + (1-\lambda)g_2; x)). \tag{12}$$

$\xi^{(2)}(g;x)$ is a monotonically decreasing function of g, so $\xi^{(2)}(g_2;x) \leq \xi^{(2)}(\lambda g_1 + (1-\lambda)g_2; x) \leq \xi^{(2)}(g_1;x)$. Hence there exists a $\lambda^* \in [0,1]$ such that

$$\xi^{(2)}(\lambda g_1 + (1-\lambda)g_2; x) = \lambda^*\xi^{(2)}(g_1;x) + (1-\lambda^*)\xi^{(2)}(g_2;x). \tag{13}$$

Being a fuzzy number, by definition $\tilde{\xi}_m$ is convex, i.e.

$$\begin{aligned} &\mu_{\tilde{\xi}_m}(\lambda^*\xi^{(2)}(g_1;x) + (1-\lambda^*)\xi^{(2)}(g_2;x)) \\ &\qquad \geq \min_{i=1,2}\{\mu_{\tilde{\xi}_m}(\xi^{(2)}(g_i;x))\}. \end{aligned} \tag{14}$$

Finally,

$$\mu_{\tilde{\xi}_m}(\xi^{(2)}(g_i;x)) = \mu_{\tilde{g}_{x;m}}(g_i), \qquad i = 1,2. \tag{15}$$

Combining (12)-(15), we obtain $\mu_{\tilde{g}_{x;m}}(\lambda g_1 + (1-\lambda)g_2) \geq \min_{i=1,2}\{\mu_{\tilde{g}_{x;m}}(g_i)\}$, which establishes convexity. (*QED*)

Because $\tilde{g}_{x;m}$ is a fuzzy number, then, as we noted in Section 2, the fuzzy average error probability P_e is a fuzzy number. Henceforth we assume that for each m, the membership function $\mu_{\tilde{\xi}_m}(\cdot)$ is symmetric about the mid-point $[\tilde{\xi}_m]_0$ and therefore by Proposition 1 the fuzzy average error probability is a fuzzy number denoted by $\tilde{P}_e$.

As shown in Theorem 1, the decision regions $\mathcal{R}_m$ are determined by the maximum of the weighted TDC indices of the fuzzy numbers $p_m\tilde{g}_{x;m}$. The TDC index of $\tilde{g}_{x;m}$ is found by applying (3) to $\tilde{g}_{x;m}$: $T(\tilde{g}_{x;m}) = \int_0^1 \frac{1}{2}([\tilde{g}_{x;m}]_1^\alpha + [\tilde{g}_{x;m}]_2^\alpha)\,d\alpha$. The mapping $g_{x;m}(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\xi)^2}{2\sigma^2})$ is symmetric about x, i.e. $g_{x;m}(\xi) = g_{x;m}(2x - \xi)$. Making use of the symmetry of the membership function $\mu_{\tilde{\xi}_m}(\xi)$ and the symmetry of the mapping $g_{x;m}(\xi)$, we obtain:

$$T(\tilde{g}_{x;m}) = \begin{cases} \frac{1}{2}\int_0^{[\tilde{\xi}_m]_0 - [\tilde{\xi}_m]_1^0}\left(G(u + |x - [\tilde{\xi}_m]_0|) + G(u - |x - [\tilde{\xi}_m]_0|)\right) \\ \quad \mu'_{\tilde{\xi}_m}([\tilde{\xi}_m]_0 - u)\,du, \text{ if } x < [\tilde{\xi}_m]_1^0 \text{ or } x > [\tilde{\xi}_m]_2^0 \\ \frac{1}{2}\left(\int_0^{[\tilde{\xi}_m]_0 - [\tilde{\xi}_m]_1^0} G(u + |x - [\tilde{\xi}_m]_0|)\mu'_{\tilde{\xi}_m}([\tilde{\xi}_m]_0 - u)\,du \right. \\ \quad + \int_0^{[\tilde{\xi}_m]_0 - x} G(u - |x - [\tilde{\xi}_m]_0|)\mu'_{\tilde{\xi}_m}([\tilde{\xi}_m]_0 - u)\,du \\ \quad \left. + \frac{1}{\sqrt{2\pi\sigma^2}}\mu_{\tilde{\xi}_m}(x)\right), \text{ if } [\tilde{\xi}_m]_1^0 \leq x \leq [\tilde{\xi}_m]_2^0 \end{cases} \tag{16}$$

where $\mu'_{\tilde{\xi}_m}(\cdot)$ is the derivative of the membership function $\mu_{\tilde{\xi}_m}(\cdot)$ and $G(a) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{a^2}{2\sigma^2})$.

The TDC index expressions given in (16) are valid for any symmetric membership function $\mu_{\tilde{\xi}_m}(\cdot)$. Now assume that the membership functions $\mu_{\tilde{\xi}_m}(\xi)$ all have the same functional form $\mu(\xi)$, but differ only in their locations on the real line. More specifically, let $\mu(\xi)$ be a given symmetric membership function centered about $\xi = 0$. Let $\Delta > 0$ be a given shift value and for $m = 1, \ldots, M$ define $\mu_{\tilde{\xi}_m}(\xi)$ such that $\mu_{\tilde{\xi}_m}(\xi + m\Delta) \stackrel{\text{def}}{=} \mu(\xi)$. For simplicity assume that the function $\mu(\xi)$ is differentiable at all points in its support except possibly at $\xi = 0$.

Proposition 2 *If the hypotheses $H_1, \ldots, H_M$ are equally likely, i.e. if $p_1 = \ldots = p_M$, and the membership functions $\{\mu_{\tilde{\xi}_m}(\xi)\}$ are defined as above, then the optimal decision regions $\mathcal{R}^*_m$ are:*

$$\mathcal{R}^*_m = \{x | \beta_{m-1} < x \leq \beta_m\}, \tag{17}$$

where the thresholds $\beta_1, \ldots, \beta_{M-1}$ are the mid-points of the intervals between consecutive fuzzy numbers: $\beta_m \stackrel{\text{def}}{=} \frac{1}{2}\left([\tilde{\xi}_m]_0 + [\tilde{\xi}_{m+1}]_0\right), m = 1, \ldots, M-1$; $\beta_0 \stackrel{\text{def}}{=} -\infty$ and $\beta_M \stackrel{\text{def}}{=} \infty$.

Proof: $T(\tilde{g}_{x;m})$ is a differentiable function of x over $(-\infty, [\tilde{\xi}_m]_0)$ and is continuous for all x. Using Leibniz' rule [7] to differentiate $T(\tilde{g}_{x;m})$, we obtain $dT(\tilde{g}_{x;m})/dx > 0$, for all $x < [\tilde{\xi}_m]_0$. In other words, for $x < [\tilde{\xi}_m]_0$, $T(\tilde{g}_{x;m})$ is a monotonically increasing function of x. The monotonicity of $T(\tilde{g}_{x;m})$ over $(-\infty, [\tilde{\xi}_m]_0)$ combined with the fact that $T(\tilde{g}_{x;m})$ is symmetric about $x = [\tilde{\xi}_m]_0$ and continuous at $x = [\tilde{\xi}_m]_0$ implies that for arbitrary m and arbitrary x, $T(\tilde{g}_{x;m})$ is a monotonocially decreasing function of $|x - [\tilde{\xi}_m]_0|$. The membership functions are shifted replicas of each other so that $T(\tilde{g}_{x;n}) = T(\tilde{g}_{x+(m-n)\Delta;m})$ for all x, m, n. Therefore,

$$\left.\begin{array}{l} T(\tilde{g}_{x;m}) > T(\tilde{g}_{x;n}) \\ \iff \quad T(\tilde{g}_{x;m}) > T(\tilde{g}_{x+(m-n)\Delta;m}) \\ \iff \quad |x - [\tilde{\xi}_m]_0| < |x + (m-n)\Delta - [\tilde{\xi}_m]_0| \end{array}\right\}, \tag{18}$$

where the second implication follows from the fact that for any given m, $T(\tilde{g}_{x;m})$ is monotonically decreasing away from $x = [\tilde{\xi}_m]_0$. Suppose that $n \leq m-1$. We will show that

$$|x - [\tilde{\xi}_m]_0| < |x + (m-n)\Delta - [\tilde{\xi}_m]_0| \iff x > \beta_{m-1}. \tag{19}$$

First, we have

$$\begin{aligned} x + \Delta - [\tilde{\xi}_m]_0 &= x + \frac{[\tilde{\xi}_m]_0 - [\tilde{\xi}_{m-1}]_0}{2} - [\tilde{\xi}_m]_0 \\ &= x - \underbrace{\frac{[\tilde{\xi}_m]_0 + [\tilde{\xi}_{m-1}]_0}{2}}_{\beta_{m-1}}. \end{aligned} \tag{20}$$

Therefore,

$$\left.\begin{array}{r} x + (m-n)\Delta - [\tilde{\xi}_m]_0 > 0, \forall n \leq m-1 \\ \iff \quad x > \beta_{m-1} \end{array}\right\}. \tag{21}$$

Second, the following inequality is satisfied for all x:

$$x - [\tilde{\xi}_m]_0 < x + (m-n)\Delta - [\tilde{\xi}_m]_0, \forall n \leq m-1. \tag{22}$$

Third, $\forall n \leq m-1$,

$$\left.\begin{array}{l} x - [\tilde{\xi}_m]_0 > -\left(x + (m-n)\Delta - [\tilde{\xi}_m]_0\right), \\ \iff \quad x - [\tilde{\xi}_m]_0 > -\frac{(m-n)\Delta}{2} \\ \iff \quad x - [\tilde{\xi}_m]_0 > -\frac{\Delta}{2} \\ \iff \quad x > \frac{[\tilde{\xi}_m]_0 + [\tilde{\xi}_{m-1}]_0}{2} = \beta_{m-1} \end{array}\right\}. \tag{23}$$

Combining (18)-(23) we conclude that $\beta_{m-1} < x$ if and only if $T(\tilde{g}_{x;m}) > T(\tilde{g}_{x;n})$, $\forall n \leq m-1$. One can also show in a similar way that $x < \beta_m$ if and only if $T(\tilde{g}_{x;m}) > T(\tilde{g}_{x;l})$, $\forall l > m$. Finally, since the point $x = \beta_m$ has zero probability measure under the $f_{X|m;\xi_m}$'s, we can assign the right end-point $x = \beta_m$ to $\mathcal{R}_m$ without changing the resulting fuzzy average error probability. (*QED*)

The thresholds $\beta_1, \ldots, \beta_{M-1}$ in (17) are independent of the form of the membership functions: as long as all hypotheses are equally likely and the membership functions are shifted versions of a given symmetric membership function, the decision rule specified in Proposition 2 is optimal. The probability of error when H_m is the true hypothesis is given by $P(E|H_m; \tilde{\xi}_m) = 1 - \left(\Phi(\frac{\beta_m - \tilde{\xi}_m}{\sigma}) - \Phi(\frac{\beta_{m-1} - \tilde{\xi}_m}{\sigma})\right)$, where $\Phi(\cdot)$ is the standard Gaussian cumulative distribution function. The function $P(E|H_m; \cdot)$ induces a fuzzy number between 0 and 1, called the fuzzy conditional probability of error given H_m. The fuzzy average error probability $\tilde{P}_e$ can be written as: $\tilde{P}_e = (1/M)\sum_{m=1}^{M} P(E|H_m; \tilde{\xi}_m)$. The α-cuts for $\tilde{P}_e$ are directly obtained from the α-cuts of the fuzzy conditional probabilities $P(E|H_m; \tilde{\xi}_m)$ as: $(\tilde{P}_e)^\alpha = (1/M)\sum_{m=1}^{M}(P(E|H_m; \tilde{\xi}_m))^\alpha$. By the resolution theorem [6], the complete set of α-cuts of a fuzzy set uniquely determines the membership function of the fuzzy set. Therefore once the α-cuts of the fuzzy conditional probabilities are computed, the membership function of the fuzzy average error probability is specified.

4. REFERENCES

[1] Lehmann, E. L., *Theory of Point Estimation,* Pacific Grove: Wadsworth, 1991.

[2] Saade, J. J., "Extension of fuzzy hypothesis testing with hybrid data," *Fuzzy Sets and Systems,*, vol. 63, no. 1, pp. 57-71, 1994.

[3] Visher, G. S., "Grain-size distribution and depositional processes," *Jour. Sediment. Petrol.*, vol. 39, no. 3, pp. 1074-1106, 1969.

[4] Nayfeh, T. H., Wheelis, W. B., Jr., and Leslie, H. D., "The fluid compensated cement log bond," *SPE paper 13044*, Society of Petroleum Engineers, Richardson, Texas, 1984.

[5] Haykin, S. *Communication Systems,* 2nd ed., New York: John Wiley & Sons, 1983.

[6] Terano, T., Asai, K., and Sugeno, M., *Fuzzy Systems Theory and Its Applications,* San Diego: Academic Press, 1992.

[7] Boas, M. L., *Mathematical Methods in the Physical Sciences,* New York: John Wiley & Sons, 1983.