ON THE PROBABILITY OF DETECTION OF A TRANSIENT SIGNAL

Chunming Han and *Peter Willett*

Electrical Engineering Dept.
Univ. of Connecticut
Storrs, CT 06269-3157
hanc(or willett)@eng2.uconn.edu

Douglas Abraham

Naval Undersea Warfare Center
New London,
Connecticut 06320
doug@next2nothing.nl.nuwc.navy.mil

ABSTRACT

The performance of Page's test for the detection of a *permanent* change in distribution is reasonably well-understood. However, there are few parallel results on its application to the detection of a *temporary* (*i.e.*, transient) change, and this is the paper's subject. Specifically, a lower bound on detectability is developed using a quantization approach; and a pair of approximations are presented, one based on a Brownian motion analogy, which yields an upper bound in the Gaussian case, and the other again on quantization. The correspondence between these and simulation appears good in both Gaussian and non-Gaussian cases with heavier tail probability.

1. INTRODUCTION

We first define the operation of Page's test [1]. With $\{Y_n\}$ (n=1,2 $\cdots$) an *iid* log-likelihood ratio sequence, define,

$$Z_n = \max\{0, \; Z_{n-1} + Y_n\}, \tag{1}$$

with $Z_0 \geq 0$. In words, Z_n is set to zero whenever it tries to go less than zero. This operation is often called *regulation.* A detection is declared the first time Z_n reaches or exceeds a threshold h, with "stopping time"

$$T = \min\{n : \; Z_n \geq h\}. \tag{2}$$

Page's test is most powerful (quickest) for the detection of a change in distribution, and, its performance as so applied is fairly well-understood (e.g. [2, 3]).

Page's procedure is also frequently applied to the detection of *transient* changes in distribution, for which problem it also possesses some optimal properties [4]; however, its performance in this application is less well-understood. Consider a Gaussian shift-in-mean transient whose strength (the distance between the means) varies as the inverse square root of the duration. Such a transient has constant SNR, meaning that the performance of a fixed-length test should be independent of the duration, and one may expect similar behavior from the Page's procedure. However, according to figure 1, this is not the case: as the duration increases, so does the detectability. It is apparent, therefore, that intuition is not sufficient, and that analysis is necessary.

THIS WORK WAS SPONSORED BY THE OFFICE OF NAVAL RESEARCH.

2. PERFORMANCE EVALUATION

Three approaches will be employed to evaluate the performance of Page's test applied for detecting a transient signal.

2.1. The Lower bound on detectability

We give a lower bound by quantizing the innovation Y_n with three levels. Consider

$$\begin{aligned} Pr\{Y_n = 1\} &= q \\ Pr\{Y_n = 0\} &= r \\ Pr\{Y_n = -1\} &= p \end{aligned} \tag{3}$$

where $q + r + p = 1$. Then, if the test starts from zero at $n = 0$, the characteristic function of T (as defined in equation (2)) is as follows [5, 6]:

$$E\{e^{-j\omega T}\} = \frac{\lambda_+ - \lambda_-}{\lambda_-^{-h}(\lambda_+ - 1) + \lambda_+^{-h}(1 - \lambda_-)} \tag{4}$$

where

$$\lambda_\pm = \frac{1 - re^{-j\omega} \pm [(1 - re^{-j\omega})^2 - 4pqe^{-2j\omega}]^{1/2}}{2pe^{-j\omega}}. \tag{5}$$

The lower bound can be obtained by taking the inverse Fourier transform of equation (4).

0-7803-2431-5/95 $4.00 © 1995 IEEE

2.2. The Brownian Motion Approximation —an Upper Bound in the Gaussian case

It can be seen that Page's test statistic is similar to a Brownian motion regulated at zero, defined as follows: Let W_t be a zero-mean unit-variance Wiener process and let

$$X_t = X_0 + \mu t + \sigma W_t \tag{6}$$

$$L_t = \max\{0, \ -\inf_{0\leq s\leq t} X_s\} \tag{7}$$

$$Z_t = X_t + L_t. \tag{8}$$

In the case where the update of Page's test is Gaussian, with the same threshold being used, the detectability of the Brownian motion is an upper bound of that of Page's test because Z_t can cross the threshold in between sample times and can be raised more in between sample times by the zero regulator than Z_n.

The characteristic function of the time-to-detection

$$T = \min\{t: \ Z_t \geq h\} \tag{9}$$

is known [7] if the Brownian motion has an initial value $Z_0 = X_0 = 0$ at time $t = 0$. Note that T is a positive real number here while it only takes integer values in the preceding section. However, the initial value of the test for a transient is distributed the same as the steady state of the test under the null hypothesis [8].

Via stochastic calculus, we found the characteristic function of the stopping time T with arbitrary initial value Z_0:

$$E\{e^{-j\omega T}\} = \frac{\psi(Z_0)}{\psi(h)} \tag{10}$$

where

$$\psi(z) = \left(\sqrt{\left(\frac{\mu}{\sigma}\right)^2 + 2j\omega} + \frac{\mu}{\sigma}\right) e^{(\sqrt{(\frac{\mu}{\sigma})^2+2j\omega}-\frac{\mu}{\sigma})\frac{z}{\sigma}} + \left(\sqrt{\left(\frac{\mu}{\sigma}\right)^2 + 2j\omega} - \frac{\mu}{\sigma}\right) e^{-(\sqrt{(\frac{\mu}{\sigma})^2+2j\omega}+\frac{\mu}{\sigma})\frac{z}{\sigma}}.$$

To derive equation (10), let $\psi(\cdot)$ denote a twice continuously differentiable function. Then $\psi(Z_t)$ is a variation finite (VF) stochastic process. Similarly, $e^{-j\omega t}$ can be regarded as a VF stochastic process, just with zero randomness. Then we have

$$e^{-j\omega t}\psi(Z_t) - e^{-j\omega 0}\psi(Z_0) = \int_0^t e^{-j\omega s} d\psi(Z_s) - j\omega \int_0^t e^{-j\omega s}\psi(Z_s)ds, \tag{11}$$

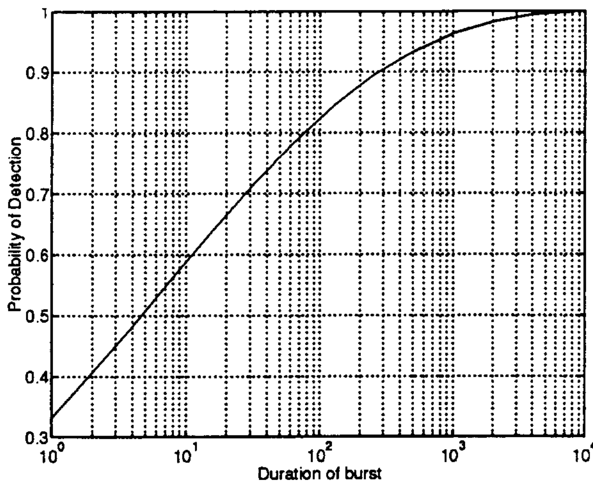

Figure 1: The probability of detection of a Gaussian shift-in-mean transient, with mean time between false alarms of 10^4.

where, of course, the integrals above are defined stochastically. Since $\psi(\cdot)$ is twice continuously differentiable, we can apply Ito's formula [9] and get

$$d\psi(Z_t) = \sigma\psi'(Z_t)dW + \frac{1}{2}\sigma^2\psi''(Z_t)dt + \mu\psi'(Z_t)dt + \psi'(0)dL_t. \tag{12}$$

Substituting equation (12) into equation (11) and taking the expectation of both sides yields

$$E\{e^{-j\omega t}\psi(Z_t)\} - \psi(Z_0) = \int_0^t e^{-j\omega s} \{[\frac{1}{2}\sigma^2\psi''(Z_s) + \mu\psi'(Z_s) - j\omega\psi(Z_s)]ds + \psi'(0)dL_s\} + \sigma E\{\int_0^t e^{-j\omega s}\psi'(Z_s)dW\}. \tag{13}$$

Note that

$$E\{\int_0^t e^{-j\omega s}\psi'(Z_s)dW\} = E\{\int_0^0 e^{-j\omega s}\psi'(Z_s)dW\} = 0 \tag{14}$$

since $\int_0^t e^{-j\omega s}\psi'(Z_s)dW$ is a local martingale.

Equation (13) is true for any twice continuously differentiable function $\psi(\cdot)$. We want to find a $\psi(\cdot)$ that zeros the right hand side of equation (13), a sufficient condition for which is

$$\frac{1}{2}\sigma^2\psi''(Z_s) + \mu\psi'(Z_s) - j\omega\psi(Z_s) = 0 \tag{15}$$

$$\psi'(0) = 0\,. \tag{16}$$

Solving the ordinary (non-stochastic) differential equation (15), subject to boundary condition (16), assuming $(\frac{\mu}{\sigma})^2 + 2j\omega \neq 0$, yields

$$\psi(z) = k\left\{\left(\sqrt{\left(\frac{\mu}{\sigma}\right)^2 + 2j\omega} + \frac{\mu}{\sigma}\right) e^{(\sqrt{(\frac{\mu}{\sigma})^2+2j\omega}-\frac{\mu}{\sigma})\frac{z}{\sigma}} + \left(\sqrt{\left(\frac{\mu}{\sigma}\right)^2 + 2j\omega} - \frac{\mu}{\sigma}\right) e^{-(\sqrt{(\frac{\mu}{\sigma})^2+2j\omega}+\frac{\mu}{\sigma})\frac{z}{\sigma}}\right\} \quad (17)$$

where k is an arbitrary constant and can be set equal to one. Substituting equation (17) into (13), and replacing t with T (notice that $Z_T = h$) results in equation (10). Again the detectability is obtained by taking the inverse Fourier transform.

2.3. Continuous-Time Moment-Matching

In another approach, the discrete-time Page's test process is approximated by a continuous-time process in which only positive and negative "jumps" of a single magnitude are allowed. The probabilities

$$\begin{aligned} Pr\{\, X_{t+\Delta t} = X_t \pm d\,\} &= \nu_{\pm}\Delta t + o(\Delta t) \\ Pr\{\, |X_{t+\Delta t} - X_t| > d\,\} &= o(\Delta t) \end{aligned} \quad (18)$$

are determined, with positive quantities d and $\nu_{\pm}$ chosen to match moments with the Page's test update under the alternative hypothesis. L_t, Z_t and T are defined the same as in equations (7), (8) and (9) respectively. If the test starts from zero at $t = 0$, then (see [5])

$$E\{e^{-j\omega T}\} = \frac{\gamma_+ - \gamma_-}{\gamma_-^{-\lceil h/d \rceil}(\gamma_+ - 1) + \gamma_+^{-\lceil h/d \rceil}(1 - \gamma_-)} \quad (19)$$

where $\lceil\cdot\rceil$ is the ceiling operation which is actually removed for a better approximation to Page's test, and

$$\gamma_{\pm} = \frac{(j\omega + \nu_+ + \nu_-) \pm \sqrt{(j\omega + \nu_+ + \nu_-)^2 - 4\nu_+\nu_-}}{2\nu_-}. \quad (20)$$

It should be noticed that if d is infinitesimal, the continuous process becomes Brownian motion.

3. EXAMPLES

First, suppose that the innovations $\{Y_n\}_{n=1}^{\infty}$ are *iid* unit-variance Gaussian random variables with a -0.2 and 0.2 shift-in-mean under the null and alternative hypotheses respectively. Suppose the desired average distance between false alarms $\bar{T}$ for the Brownian motion is 10^4 so that the average distance between false alarms for the Page's test is over 10^4. Taking the derivative of equation (10) with respect to $(-j\omega)$ and setting $\omega = 0$, we get

$$\bar{T} = \frac{\sigma^2}{2\mu^2}\left(\frac{2\mu(h - Z_0)}{\sigma^2} + e^{-\frac{2\mu h}{\sigma^2}} - e^{-\frac{2\mu Z_0}{\sigma^2}}\right). \quad (21)$$

Let $\bar{T} = 10^4, \mu = -0.2, \sigma = 1$ and $Z_0 = 0$ for simplicity. Then the threshold solved from equation (21) is approximately $h = 17$. Then the average distance between false alarms, in which case the shift-in-mean is -0.2, for the Page's test is 1.5×10^4 according to simulation. While the simulation result of the probability of detection, in which case the shift-in-mean is 0.2, is shown (the solid line) in figure 2 together with its upper bound obtained through Brownian motion.

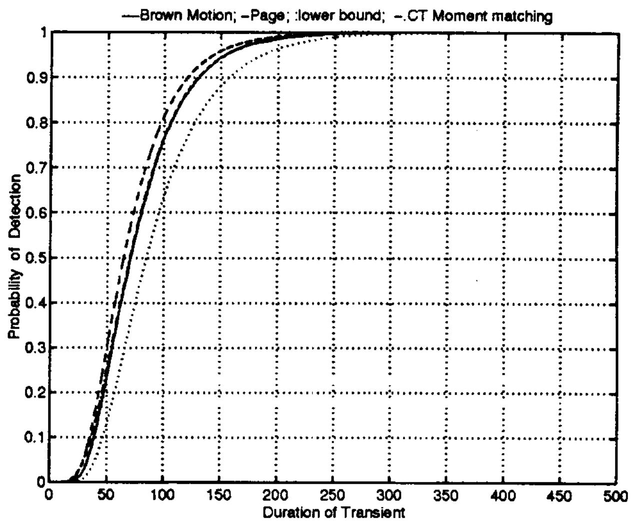

Figure 2: The detectability of a Gaussian transient .

In order to get the p, q, and r in equation (3), the innovation needs to be quantized. Suppose we quantize the *pdf* under the null hypothesis with thresholds ± 0.359. We will get $p_n = 0.4368$, $r_n = 0.2751$ and $q_n = 0.2881$. Quantizing the *pdf* under the alternative hypothesis with the same threshold yields the probabilities $p_a = 0.2881$, $r_a = 0.2751$ and $q_a = 0.4368$.

To find the threshold h needed in equation (4) meeting the requirement of the mean time between false alarms. First we take the derivative of equation (4) with respect to $(-j\omega)$ and set $\omega = 0$. We get

$$\bar{T} = \frac{1}{q_n - p_n}\left[\frac{1 - (p_n/q_n)^h}{p_n/q_n - 1} - h\right]. \quad (22)$$

Therefore h can be solved for the desired $\bar{T} = 1.5 \times 10^4$. Then with the h so obtained, we can get the characteristic function of T under the alternative hypothesis by substituting the p, q and r in equation (5) with p_a, q_a and r_a respectively. Then we take the inverse Fourier transform numerically to get the probabilities for the stopping time. Taking the cumulative sum of these probabilities gives the probability of detection which varies as the quantization thresholds chosen. The thresholds that maximize the probability of detection are ± 0.359. The corresponding tightest lower bound of the probability of detection given by the three-level quantizer is shown in figure 2 (the lower dotted curve).

To moment-match Page's test with a continuous time process as presented in section 2.3 the shift-in-mean equation,

$$(\nu_+ - \nu_-)d = \mu, \tag{23}$$

and the variance equation,

$$(\nu_+ + \nu_-)d^2 = \sigma^2, \tag{24}$$

have to be solved simultaneously after d is determined. Heuristically, if positive solutions for $\nu_\pm$ can be ensured, two standard deviation jumps is about right:

$$d = 2\sigma + \mu. \tag{25}$$

In this example, the solutions $\nu_+ = 0.1488$ and $\nu_- = 0.0579$ are substituted into equation (20), and the corresponding characteristic equation is obtained.

In each of the three cases, namely three-level quantization, Brownian motion and continuous-time moment-matching, we obtained the probability of detection via the characteristic function of the stopping time. In the three-level quantization case, the characteristic function is a discrete time Fourier transform, the inverse transform of which is the probability point mass for the time to detection T. While in the other two cases, the characteristic function is a continuous time Fourier transform, the inverse transform of which is the probability density function of T.

Figure 3 corresponds to figure 2 except here a cell average constant false alarm rate (CA-CFAR) test is investigated where heavy non-Gaussian probability tails are involved.

4. CONCLUSION

The Brownian motion gives a relatively tight upper bound in the Gaussian case. It also tends to be an upper bound in non-Gaussian cases. The quantization approach gives a genuine lower bound in any case. It usually becomes less tight in non-Gaussian cases with heavier tail probability. The moment-matching is very accurate, even in non-Gaussian cases. In most cases of interest, using the three computationally inexpensive approaches together, a confident, accurate assessment of the performance can be obtained.

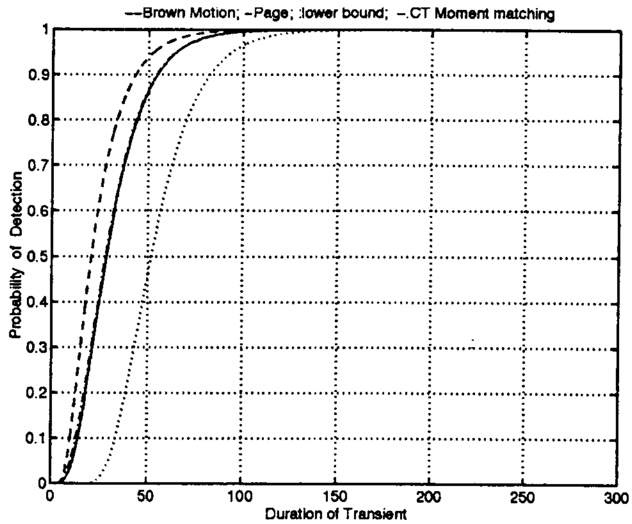

Figure 3: This is the same as figure (2) except that the test is for a CA-CFAR transient.

5. REFERENCES

[1] Page, E. S., "Continuous inspection schemes," *Biometrika*, vol. 41, pp. 100-115, 1954.

[2] Lorden, G., "Procedures for reacting to a change in distribution," *Ann. Math. Statist.*, vol. 42, pp. 1897-1908, 1971.

[3] Basseville, M. and Nikiforov, I. V., *Detection of Abrupt Changes.* Prentice Hall, 1993.

[4] Broder, B. and Schwartz, S., *Quickest Detection Procedures and Transient Signal Detection*, ONR Technical Report 21, Nov. 1990.

[5] Macken, C. A. and Taylor, H. M., "On deviation from the maximum in a stochastic process," *SIAM J. Appl. Math.*, vol. 32, pp. 96-104, Jan. 1977.

[6] Basseville, M., "Edge detection using sequential methods for change in level-part II: sequential detection of change in mean," *IEEE Trans. on ASSP*, vol. 29, pp. 32-50, 1981.

[7] Taylor, H. M., "A stopped Brownian motion formula," *Ann. Prob.*, vol. 3, pp. 234-246, Feb. 1975.

[8] Han, C., Willett, P. and Abraham, D., *Some Bounds and Approximations for the Probability of Detection of a Transient Signal.* The University of Connecticut, Technical Report TR-94-4, 1994.

[9] Ito, K., *Diffusion processes and their sample paths.* Springer-Verlag, 1965.