

MAXIMUM SINR BEAMFORMING FOR CORRELATED SOURCES

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ABSTRACT

In this paper, we consider the signal-to-interference plus noise ratio (SINR) performance of several beamforming algorithms, taking particular account of the contribution of sources correlated with the desired signal. In addition, we derive an optimal method that maximizes SINR by combining with the desired signal estimate any components of the interference/multipaths that are correlated with it. To facilitate performance comparisons, we will only consider the case where the signal directions of arrival (DOAs) are precisely known. The extension to unknown DOAs is straightforward. Our analysis includes the first order effects of array calibration errors, and is verified by numerical simulation.

1. INTRODUCTION

The problem of beamforming in situations where the received signals are correlated has been studied by a number of researchers. For example, it is well known that the standard linearly constrained beamformers (*e.g.*, see [1]) exhibit a degree of signal cancellation in such scenarios. A variety of *ad hoc* techniques have been proposed to mitigate this effect, ranging from subtractive preprocessors to methods that decorrelate the desired signal and interference. The general philosophy in most approaches such as these is that the beamformer weights must cancel the contributions from interfering signals, even if they are correlated with the signal of interest (SOI).

On the other hand, beamformer weights derived from a minimum mean squared error (MMSE) criterion (*i.e.*, the Wiener solution) provide an estimate that is as close as possible to the actual signal, regardless of whether this nulls out the interferences or not. It makes sense that if one of the interfering signals is 100% correlated with the SOI (due, for example, to multipath propagation), performance can be improved by optimally combining it with the "direct path" signal. This is the approach taken in [2], where the MMSE weights

are approximated from the data for the special case of a uniform linear array. A similar technique for general array structures was presented in [3] (referred to herein as the *structured stochastic estimator*, or SSE), although it requires DOA estimates for the SOI and all interferences.

One of the goals of this paper is the development of an optimal method for estimating the SOI by combining the contributions of correlated signals, but using maximum SINR rather than MMSE as the performance metric. In the course of this development, we analyze the SINR performance of a number of other algorithms, including SSE, standard least squares (LS), total least squares (TLS) [4], and the principal components (PC) method (*e.g.*, [5, 6]). As in the MSE analysis of [7], a simple array perturbation model is used to incorporate the effects of array calibration errors into the analysis. The SINR performance of the LS algorithm has been studied in [8], but contributions from signals correlated with the SOI were not taken into account. Before moving on the analysis, we briefly describe our assumed data model in the next section.

2. DATA MODEL AND ASSUMPTIONS

Assume that d narrowband signals impinge on an array of m sensors. The output of the array $\mathbf{x}(t)$ is assumed to be a superposition of the d signals corrupted by zero-mean, spatially and temporally white noise $\mathbf{n}(t)$:

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t) \quad t = 1, \dots, N$$

where $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_d)]$ is the collection of sensor responses parameterized by the DOAs $\boldsymbol{\theta}$, and $\mathbf{s}(t) = [s_1(t), \dots, s_d(t)]^T$ is a vector containing the complex envelope of all point sources, including the SOI and any interferers. Without loss of generality, we assume that $s_d(t)$ is the SOI, and $s_1(t), \dots, s_{d-1}(t)$ are "interferers". We separate the interference into two components, one correlated with the desired signal, and

the other not:

$$\mathbf{s}(t) = \begin{bmatrix} \tilde{\mathbf{s}}(t) \\ s_d(t) \end{bmatrix} = \begin{bmatrix} \mathbf{s}^\perp(t) + \frac{1}{\sigma_d^2} \mathbf{r} s_d(t) \\ s_d(t) \end{bmatrix}, \quad (1)$$

where $\mathcal{E}\{\mathbf{s}^\perp(t)\mathbf{s}_d^*(t)\} = \mathbf{0}$, \mathbf{r} is defined as the first $d-1$ rows of the last column of \mathbf{R}_{ss} :

$$\mathbf{R}_{ss} = \mathcal{E}\{\mathbf{s}(t)\mathbf{s}^*(t)\} = \begin{bmatrix} \mathbf{R}_{\tilde{s}\tilde{s}} & \mathbf{r} \\ \mathbf{r}^* & \sigma_d^2 \end{bmatrix}, \quad (2)$$

and σ_d^2 is the power of the SOI. The covariance of $\mathbf{s}^\perp(t)$ is easily shown to be

$$\mathbf{R}_{s_s}^\perp = \mathcal{E}\{\mathbf{s}^\perp(t)\mathbf{s}^{\perp*}(t)\} = \mathbf{R}_{\tilde{s}\tilde{s}} - \mathbf{r}\mathbf{r}^*/\sigma_d^2. \quad (3)$$

It is also convenient to separate the array response into two parts, as follows:

$$\mathbf{A}(\theta) = [\bar{\mathbf{A}} \quad \mathbf{a}(\theta_d)].$$

Because of various unavoidable errors (mutual coupling, gain/phase perturbations, *etc.*), $\mathbf{A}(\theta)$ will be different from the available array calibration model $\mathbf{A}_0(\theta)$:

$$\mathbf{A}(\theta) = \mathbf{A}_0(\theta) + \tilde{\mathbf{A}}(\theta).$$

Similarly, $\bar{\mathbf{A}}_0$ and $\mathbf{a}_0(\theta_d)$ will denote the nominal calibrated values for $\bar{\mathbf{A}}$ and $\mathbf{a}(\theta_d)$, respectively. To compare in a rough sense the effects of such calibration errors, we will assume in our analysis that the columns of the perturbation matrix $\tilde{\mathbf{A}}$ are random with zero mean and second-order moments

$$\mathcal{E}\{\tilde{\mathbf{a}}(\theta_i)\tilde{\mathbf{a}}^*(\theta_j)\} = \sigma_a^2 \mathbf{I} \delta_{i,j} \quad (4)$$

$$\mathcal{E}\{\tilde{\mathbf{a}}(\theta_i)\tilde{\mathbf{a}}^T(\theta_j)\} = 0, \quad (5)$$

where $\delta_{i,j}$ is the Kronecker delta.

3. SINR ANALYSIS

For a given set of beamformer weights \mathbf{w} , the SOI estimate can be written as

$$\begin{aligned} \hat{s}_d(t) &= \mathbf{w}^* \mathbf{x}(t) \\ &= \mathbf{w}^* \left(\frac{1}{\sigma_d^2} \bar{\mathbf{A}} \mathbf{r} + \mathbf{a}(\theta_d) \right) s_d(t) \\ &\quad + \mathbf{w}^* \bar{\mathbf{A}} \mathbf{s}^\perp(t) + \mathbf{w}^* \mathbf{n}(t). \end{aligned} \quad (6)$$

The estimate $\hat{s}_d(t)$ can thus be decomposed into the following three components:

- signal component

$$\mathbf{c}_S = \mathbf{w}^* \left(\frac{1}{\sigma_d^2} \bar{\mathbf{A}} \mathbf{r} + \mathbf{a}(\theta_d) \right) s_d(t), \quad (7)$$

- interference component

$$\mathbf{c}_I = \mathbf{w}^* \bar{\mathbf{A}} \mathbf{s}^\perp(t), \quad (8)$$

- noise component

$$\mathbf{c}_N = \mathbf{w}^* \mathbf{n}(t). \quad (9)$$

The SINR then is defined to be

$$\text{SINR} = \frac{\mathcal{E}\|\mathbf{c}_S\|^2}{\mathcal{E}\|\mathbf{c}_I\|^2 + \mathcal{E}\|\mathbf{c}_N\|^2}. \quad (10)$$

where $\mathcal{E}\|\mathbf{c}_S\|^2$, $\mathcal{E}\|\mathbf{c}_I\|^2$, and $\mathcal{E}\|\mathbf{c}_N\|^2$ are respectively the power of desired signal, interference, and noise in $\hat{s}_d(t)$. Note that the expectation is also taken with respect to any random array errors present, such as those in (4)-(5). The SINR performance of any given beamforming algorithm may be determined by substituting the appropriate \mathbf{w} into (7)-(9), and then evaluating the expectations in (10).

3.1. A Comparison of LS and TLS

As an example of the insight that can be gained from such an analysis, we compare the SINR performance of the LS and TLS beamformers.

It is shown in [8] that the LS weight vector for the d^{th} signal may be written as

$$\mathbf{w}_{LS} = \mathbf{P}_{\bar{\mathbf{A}}_0}^\perp \mathbf{a}_{d0} (\mathbf{a}_{d0}^* \mathbf{P}_{\bar{\mathbf{A}}_0}^\perp \mathbf{a}_{d0})^{-1}, \quad (11)$$

where $\mathbf{P}_{\bar{\mathbf{A}}_0}^\perp = \mathbf{I} - \bar{\mathbf{A}}_0 (\bar{\mathbf{A}}_0^* \bar{\mathbf{A}}_0)^{-1} \bar{\mathbf{A}}_0^*$. Substituting (11) into (7)-(9), it can be shown that

$$\mathcal{E}\|\mathbf{c}_S\|^2 = N \mathbf{w}_{LS}^* [\mathbf{A}_0 \mathbf{R}_d \mathbf{A}_0^* + \sigma_a^2 \text{Tr}(\mathbf{R}_d) \mathbf{I}] \mathbf{w}_{LS} \quad (12)$$

$$\mathcal{E}\|\mathbf{c}_I\|^2 = N \mathbf{w}_{LS}^* [\bar{\mathbf{A}}_0 \mathbf{R}_{ss}^\perp \bar{\mathbf{A}}_0^* + \sigma_a^2 \text{Tr}(\mathbf{R}_{ss}^\perp) \mathbf{I}] \mathbf{w}_{LS} \quad (13)$$

$$\mathcal{E}\|\mathbf{c}_N\|^2 = N \sigma_n^2 \mathbf{w}_{LS}^* \mathbf{w}_{LS} = \frac{N \sigma_n^2}{\mathbf{a}_{d0}^* \mathbf{P}_{\bar{\mathbf{A}}_0}^\perp \mathbf{a}_{d0}}. \quad (14)$$

After some algebraic manipulation, the LS output SINR is shown to be

$$\text{SINR}_{LS} \simeq \frac{\sigma_d^2 \mathbf{a}_{d0}^* \mathbf{P}_{\bar{\mathbf{A}}_0}^\perp \mathbf{a}_{d0} + \sigma_a^2 \text{Tr}(\mathbf{R}_d)}{\sigma_a^2 \text{Tr}(\mathbf{R}_{ss}^\perp) + \sigma_n^2}.$$

It is shown in [4, 9] that for large N , the TLS beamformer yields

$$\hat{s}_{TLS}(t) = (\mathbf{A}_0^* \mathbf{P}_A \mathbf{A}_0)^{-1} \mathbf{A}_0^* \mathbf{P}_A [\mathbf{A} \mathbf{s}(t) + \mathbf{n}(t)].$$

To first order

$$\mathbf{A}_0^* \mathbf{P}_A \mathbf{A}_0 = \mathbf{A}_0^* \mathbf{A}_0 - \mathbf{A}_0^* \mathbf{P}_A^\perp \mathbf{A}_0 \simeq \mathbf{A}_0^* \mathbf{A}_0,$$

and thus

$$\hat{\mathbf{s}}_{TLS}(t) \simeq \mathbf{A}_0^\dagger \mathbf{A} \mathbf{s}(t) + \mathbf{A}_0^\dagger \mathbf{P}_A \mathbf{n}(t),$$

where $\mathbf{A}_0^\dagger = (\mathbf{A}_0^* \mathbf{A}_0)^{-1} \mathbf{A}_0^*$. Notice that the first term is the same as LS. Hence, the TLS signal and interference power are the same as that of LS in (12)-(13). To obtain the noise power, we observe that since

$$\begin{aligned} \mathbf{A}_0^\dagger \mathbf{P}_A &= (\mathbf{A}_0^* \mathbf{A}_0)^{-1} \mathbf{A}_0^* - (\mathbf{A}_0^* \mathbf{A}_0)^{-1} \mathbf{A}_0^* \mathbf{P}_A^\perp \\ &\simeq (\mathbf{A}_0^* \mathbf{A}_0)^{-1} \mathbf{A}_0^* + (\mathbf{A}_0^* \mathbf{A}_0)^{-1} \tilde{\mathbf{A}}^* \mathbf{P}_{\tilde{\mathbf{A}}_0}^\perp, \end{aligned}$$

and

$$\begin{aligned} (\mathbf{A}_0^* \mathbf{A}_0)^{-1} &= \frac{1}{\mathbf{a}_{d0}^* \mathbf{P}_{\tilde{\mathbf{A}}_0}^\perp \mathbf{a}_{d0}} \times \\ &\left[\begin{array}{cc} (\tilde{\mathbf{A}}_0^* \tilde{\mathbf{A}}_0)^{-1} \mathbf{a}_{d0}^* \mathbf{P}_{\tilde{\mathbf{A}}_0}^\perp \mathbf{a}_{d0} + \tilde{\mathbf{A}}_0^\dagger \mathbf{a}_{d0} \mathbf{a}_{d0}^* \tilde{\mathbf{A}}_0^{\dagger *} & -\tilde{\mathbf{A}}_0^\dagger \mathbf{a}_{d0} \\ -\mathbf{a}_{d0}^* \tilde{\mathbf{A}}_0^{\dagger *} & 1 \end{array} \right], \end{aligned}$$

the TLS noise component is given by

$$\mathbf{c}_N \simeq \frac{1}{\mathbf{a}_{d0}^* \mathbf{P}_{\tilde{\mathbf{A}}_0}^\perp \mathbf{a}_{d0}} \left[\mathbf{a}_{d0}^* + \tilde{\mathbf{a}}_d^* - \mathbf{a}_{d0}^* \tilde{\mathbf{A}}_0^{\dagger *} \tilde{\mathbf{A}}^* \right] \mathbf{P}_{\tilde{\mathbf{A}}_0}^\perp \mathbf{n}(t).$$

The associated noise power is thus

$$\begin{aligned} \mathcal{E} \|\mathbf{c}_N\|^2 &= N \sigma_n^2 \left[\frac{1}{\mathbf{a}_{d0}^* \mathbf{P}_{\tilde{\mathbf{A}}_0}^\perp \mathbf{a}_{d0}} + \right. \\ &\left. \frac{\sigma_a^2 (m-d) [\mathbf{a}_{d0}^* \tilde{\mathbf{A}}_0 (\tilde{\mathbf{A}}_0^* \tilde{\mathbf{A}}_0)^{-2} \tilde{\mathbf{A}}_0^* \mathbf{a}_{d0} + 1]}{(\mathbf{a}_{d0}^* \mathbf{P}_{\tilde{\mathbf{A}}_0}^\perp \mathbf{a}_{d0})^2} \right]. \end{aligned} \quad (15)$$

Compared to (14), (15) has an extra positive term. Hence, although TLS attempts to take the array errors into account in a reasonable way, it yields a lower output SINR than LS. The inferior performance of TLS relative to LS is also shown using the MSE performance metric in [7, 9]. This is very interesting especially in light of other work which indicates just the opposite. For example, in [10] it is shown that for linear equations of the form $\mathbf{A}_0 \mathbf{S} = \mathbf{X}$ where the errors on \mathbf{A}_0 and \mathbf{X} are independent and identically distributed, TLS asymptotically outperforms LS when $m \gg N$. However, for the problem we are considering, the relationship is just the opposite: $N \gg m$. Whereas the number of parameters to be estimated remains fixed in the analysis of [10], here this number is asymptotically growing.

The analysis of other algorithms such as SSE and PC are somewhat more involved, primarily because their weight vectors depend on data-derived random quantities even in the known DOA case. The performance of these algorithms is investigated in [11], where among other results it is shown that the PC method is very sensitive to source correlation.

3.2. A Maximum SINR Beamformer

We turn now to the question of choosing a set of weights that maximize SINR, as defined in (10). If we treat \mathbf{w} as a deterministic quantity, substituting (7)-(9) into (10) and using the simple array error model of (4)-(5) yields

$$\text{SINR} = \frac{\mathbf{w}^* [\mathbf{A}_0 \mathbf{R}_d \mathbf{A}_0^* + \sigma_a^2 \text{Tr}(\mathbf{R}_d) \mathbf{I}] \mathbf{w}}{\mathbf{w}^* [\mathbf{B}_0 \mathbf{R}_{ss}^\perp \mathbf{B}_0^* + (\sigma_a^2 \text{Tr}(\mathbf{R}_{ss}^\perp) + \sigma_n^2) \mathbf{I}] \mathbf{w}}, \quad (16)$$

where \mathbf{R}_d is defined as

$$\mathbf{R}_d = \begin{bmatrix} \frac{1}{\sigma_d^2} \mathbf{r} \mathbf{r}^* & \mathbf{r} \\ \mathbf{r}^* & \sigma_d^2 \end{bmatrix}.$$

The optimal weight vector that maximizes SINR is thus the eigenvector that corresponds to the largest eigenvalue λ_{\max} of the following generalized eigendecomposition:

$$[\mathbf{A}_0 \mathbf{R}_d \mathbf{A}_0^* + \sigma_a^2 \text{Tr}(\mathbf{R}_d) \mathbf{I}] \mathbf{w}_{OPT} =$$

$$\lambda_{\max} [\tilde{\mathbf{A}}_0 \mathbf{R}_{ss}^\perp \tilde{\mathbf{A}}_0^* + (\sigma_a^2 \text{Tr}(\mathbf{R}_{ss}^\perp) + \sigma_n^2) \mathbf{I}] \mathbf{w}_{OPT}. \quad (17)$$

and the resulting SINR is given by

$$\text{SINR}_{OPT} = \lambda_{\max}. \quad (18)$$

Of course, the quantities \mathbf{A}_0 , \mathbf{R}_d , σ_n^2 , and \mathbf{R}_{ss}^\perp used above are not known *a priori*, and consequently must be estimated from the data. This is easily done given the DOAs or estimates of them, and is a standard procedure in all DF-based beamforming algorithms like LCMV, LS, *etc.* The performance advantage of the optimal method is demonstrated in the next section by means of some simulation examples.

4. SIMULATION EXAMPLES

In our first example, we consider a case where two equal-powered uncorrelated random gaussian signals with correlation coefficient $\rho = 0.9$ impinge on a 6-sensor uniform linear array (ULA) with an SNR of 10dB. Zero-mean white gaussian noise was assumed. The DOA of the first signal was fixed at 0° . The DOA of the second signal was varied between 2° and 20° , and the DOAs of both signals were assumed known. The array response was perturbed according to (4)-(5), with $\sigma_a = 0.2$. A total of 100 trials were conducted at each DOA separation with 500 snapshots per trial, and the resulting SINR for PC, LS, TLS, SSE, and the optimal method are plotted in Figure 1. Figure 1 shows that the optimal method and SSE yield the highest SINR among all the methods, with the optimal method performing slightly better. The PC method is very sensitive to source correlation, and has a very low output

SINR at every DOA separation. The output SINR of LS is about 9dB less than the optimal method and about 3dB higher than TLS at a DOA separation of 2° . Note that the difference between all the algorithms except PC becomes negligible as the DOA separation becomes larger.

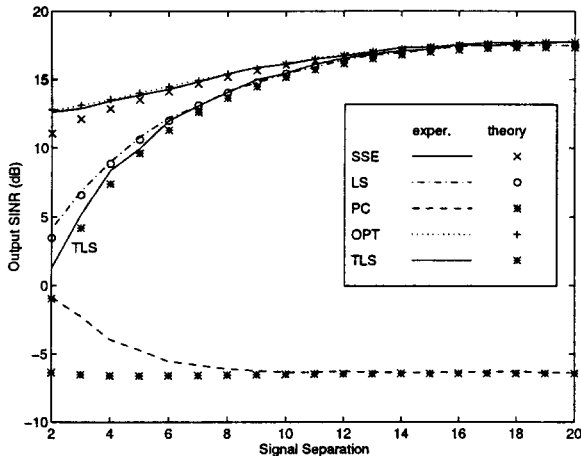


Figure 1: SINR Performance of Various Beamforming Methods vs. DOA Separation; $\rho = 0.9$, Known DOAs and Random Signals.

Our second example considers a two-ray multipath channel where the impulse response from the transmitter to the reference antenna is given by

$$p(n) + 0.9e^{-j\pi/2}p(n),$$

and where $p(n)$ is a Nyquist-shaped pulse with 65% excess bandwidth truncated to six symbol periods. A BPSK signal with 10dB SNR passes through this channel and is sampled twice per symbol at the array. The DOA of the main path (the one with unity gain) is fixed at 0° and the DOA of the multipath is varied between 5° and 20° . The array response was perturbed according to (4)-(5), with $\sigma_a = 0.2$, and the DOAs were estimated using the method described in [12]. The SINR of LS, SSE, and the optimal method based on an average of 250 trials at each simulation point is plotted in Figure 2. The improvement in performance offered by the optimal approach is clearly evident in this example, and is as much as 3-4 dB in the difficult cases.

5. REFERENCES

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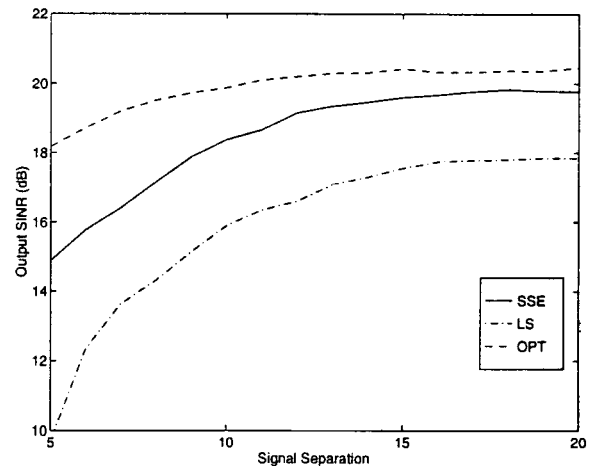


Figure 2: SINR Performance of SSE, LS, and Optimal Method vs. DOA Separation; Unknown DOAs and BPSK Signals.

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