ANALYSIS OF SIGNAL ESTIMATION USING UNCALIBRATED ARRAYS

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ABSTRACT

We consider the problem of separating and estimating the waveforms of superimposed signals received using an array of uncalibrated sensors. The array elements are assumed to have the same unknown gain pattern, up to an unknown multiplicative factor. The phases of the elements are arbitrary and unknown. In this paper we analyze the quality of the estimated signal in terms of the output signal to interference ratio (SIRO) and output signal to noise ratio (SNRO). It is shown that uncalibrated arrays can be used successfully for signal separation and estimation using only second order moments. The analysis is verified by Monte Carlo experiments using an algorithm for steering vector estimation presented in [1].

1. INTRODUCTION

The problem of separation and reconstruction of superimposed signals using an array of sensors attracted considerable interest in the last decade. The increasing use of cellular communications is expected to present a higher demand for effective methods for interference canceling and signal separation.

Most of the work on this problem [2],[3], [4], [5] concentrated on the case where the array manifold is known, *i.e.*, where the array is well calibrated. In practice, it is difficult to maintain a precisely calibrated array. Temperature, pressure, humidity, mechanical vibrations and objects in the near field all affect the calibration precision. Furthermore, the presence of multipath changes the response of the array to the signal arriving from a given source, in complicated and unpredictable ways. There is, therefore, considerable practical interest in the development of signal estimation techniques which are able to operate with uncalibrated arrays.

It is known that "blind" estimation is possible for non-Gaussian signals, by using high order statistics of the received data [6, 7, 8]. The term "blind" is used to indicate that the array manifold is assumed to be completely unknown. Techniques based on second-order statistics can not solve the "blind" estimation problem. In other words, the assumption of non-Gaussianity is essential for this approach.

It is interesting to note, however, that if we assume that the array element have the same unknown gain pattern, up to an unknown multiplicative factor, signal estimation becomes possible using only second order information. This is a practical assumption, since the gain pattern of antennas is much more stable than their phase pattern. In other words, while "blind" estimation requires non-Gaussian signals and high-order statistics, "almost blind" estimation is possible for any type of signal, using second order statistics. Such algorithms were recently presented in [1] and [9].

The focus of this paper is on the performance analysis of "almost blind" signal estimation, in the sense described above. The analysis is carried out for signals which are Gaussian, narrowband, and statistically uncorrelated. The quality of the estimated signal is defined in terms of the output signal to interference ratio (SIRO) and output signal to noise ratio (SNRO). Closed form expressions are derived for the SIRO and SNRO of an uncalibrated array, using the Cramer Rao Bound (CRB) on the variance of the estimated steering vectors. By evaluating these expressions for selected test cases it is shown that uncalibrated arrays can be used successfully for signal separation and estimation using only second order moments. To validate our analysis we compared the theoretical error variances to experimental variances obtained by performing signal estimation with a particular algorithm for steering vector estimation. The algorithm used was the one presented in [1].

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2. PROBLEM FORMULATION

We begin by describing the data model for the observation of narrowband signals by an array of sensors. We consider an M-element array of sensors and N narrowband signal sources, and define the $M \times 1$ vector $\mathbf{a}_n$ to be the complex array response for the nth source.

The outputs of the M array elements at the k-th sample are arranged in an $M \times 1$ vector,

$$\mathbf{x}(k) = \mathbf{A}\mathbf{s}(k) + \mathbf{u}(k) \qquad k = 1, 2 \cdots N_s; \tag{1}$$

where $\mathbf{u}(k)$ is the noise vector, $\mathbf{s}(k)$ is the signal vector, and

$$\mathbf{A} \triangleq [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_N] \tag{2}$$

Assuming that the signal vectors $\mathbf{s}(k)$ and the noise vectors $\mathbf{u}(k)$ are realizations of stationary, zero mean Gaussian random processes, and that there is no correlation between the different signals and no correlation between the noise and the signals, the data covariance matrix is

$$\mathbf{R} \triangleq \mathrm{E}\{\mathbf{x}(k)\mathbf{x}^H(k)\} = \mathbf{A}\mathbf{P}\mathbf{A}^H + \eta\mathbf{I} \tag{3}$$

where $\mathbf{P}$ is the signal covariance matrix (a diagonal matrix) and $\eta\mathbf{I}$ is the noise covariance matrix.

In this work we focus on the case where the array sensors are uncalibrated. We assume that the uncalibrated sensors have the same unknown gain pattern, up to an unknown multiplicative factor. The phases of the elements are arbitrary and unknown. We therefore use the following model for $\mathbf{A}$.

$$\mathbf{A} = \mathbf{G}\mathbf{C}\mathbf{H} \tag{4}$$

$$[\mathbf{G}]_{pq} = \begin{cases} g_p & \text{for } p = q \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$[\mathbf{H}]_{pq} = \begin{cases} h_p & \text{for } p = q \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$[\mathbf{C}]_{pq} = e^{j\phi_{pq}} \tag{7}$$

The positive real numbers $g_1, g_2 \cdots g_M$ are the unknown multiplicative factors of the sensor gain patterns. The diagonal matrix $\mathbf{H}$ represents the gain variation as a function of direction. Since $\mathbf{H}$ only affects the received signals power, see (3), it can not be separated from $\mathbf{P}$ and therefore we set $\mathbf{H} = \mathbf{I}$, without loss of generality. The constants ϕ_{pq} are the unknown sensor phase responses. We are interested in estimating the signal samples $\mathbf{s}(k)$. The signals may be estimated by first estimating the array steering vectors, as will be shown shortly. Note that the observations, namely $\mathbf{x}(k)$, do not change if $\mathbf{A}$ is right multiplied by a diagonal matrix while $\mathbf{s}(k)$ is left multiplied by the inverse of the same diagonal matrix. This means that the steering vectors and the signals can be observed (and estimated) only up to a multiplicative complex scalar. We therefore assume, without loss of generality, that the first element of each steering vector is one.

3. SIGNAL ESTIMATION

Perhaps the best known method for estimating the signals $\mathbf{s}(k)$ [2] is by first obtaining an estimate of the steering vectors matrix, $\hat{\mathbf{A}}$, and then evaluating the relation

$$\hat{\mathbf{s}}(k) = (\hat{\mathbf{A}}^H\hat{\mathbf{A}})^{-1}\hat{\mathbf{A}}^H\mathbf{x}(k) \tag{8}$$

The signal estimates are affected by the imperfection of the steering vector estimation and by the noise and the presence of multiple signals.

Next we obtain expressions for the quality of the signal estimation under these conditions. Without loss of generality we let the N-th signal be the desired signal and the other signals being the undesired or interfering signals.

We make the following partition of $\mathbf{A}$,

$$\mathbf{A} = [\mathbf{D}, \mathbf{v}] \tag{9}$$

where $\mathbf{v}$ stands for the last column of $\mathbf{A}$ and $\mathbf{D}$ stands for the $M \times N-1$ matrix associated with the first $N-1$ columns of $\mathbf{A}$. We define the matrices,

$$\mathbf{T}_1 \triangleq \mathbf{D}(\mathbf{D}^H\mathbf{D})^{-1}\mathbf{D}^H \tag{10}$$

$$\mathbf{T} \triangleq \mathbf{I} - \mathbf{T}_1 \tag{11}$$

$$\mathcal{D} \triangleq (\mathbf{D}^H\mathbf{D})^{-1}\mathbf{D}^H \tag{12}$$

where $\mathbf{T}_1$ is the projection matrix on the column space of $\mathbf{D}$, the matrix $\mathbf{T}$ is the projection matrix on the null space of $\mathbf{D}^H$, and $\mathcal{D}$ is the left pseudo-inverse of $\mathbf{D}$. Using these definitions it can be shown that

$$(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H = \frac{1}{\mathbf{v}^H\mathbf{T}\mathbf{v}}\begin{bmatrix} \mathcal{D}[(\mathbf{v}^H\mathbf{T}\mathbf{v})\mathbf{I} - \mathbf{v}\mathbf{v}^H\mathbf{T}] \\ \mathbf{v}^H\mathbf{T} \end{bmatrix} \tag{13}$$

For *noiseless observation* and *perfect* estimation of the steering vectors (8) and (13) yield

$$\begin{aligned} \hat{s}_N(k) = [\hat{\mathbf{s}}(k)]_N &= \frac{\mathbf{v}^H\mathbf{T}\mathbf{x}(k)}{\mathbf{v}^H\mathbf{T}\mathbf{v}} = \frac{\mathbf{v}^H\mathbf{T}\mathbf{A}\mathbf{s}(k)}{\mathbf{v}^H\mathbf{T}\mathbf{v}} \\ &= \frac{\mathbf{v}^H[0, \mathbf{T}\mathbf{v}]\mathbf{s}(k)}{\mathbf{v}^H\mathbf{T}\mathbf{v}} = s_N(k) \end{aligned} \tag{14}$$

which is the desired result. The last equation indicates that the least squares estimator of the signal involves two operations:

i) Projection of the observation vector on the null space associated with the first $(N-1)$ signals (*i.e.*, multiplication of $\mathbf{x}(k)$ by $\mathbf{T}$.) The result is $\mathbf{Tv}s_N(k)$. This operation isolates the N-th signal by elimination of all the other signals. It may also be viewed as steering nulls of the array radiation pattern towards the first $N-1$ sources.

ii) Estimation of $s_N(k)$ by adding up the desired signal components (14).

In practice, the measurements are noisy and the estimation of steering vectors is not perfect. In this case we obtain

$$\begin{aligned}\hat{s}_N(k) &= \frac{\hat{\mathbf{v}}^H\hat{\mathbf{T}}\mathbf{x}(k)}{\hat{\mathbf{v}}^H\hat{\mathbf{T}}\hat{\mathbf{v}}} = \frac{\hat{\mathbf{v}}^H\hat{\mathbf{T}}[\mathbf{A}\mathbf{s}(k)+\mathbf{u}(k)]}{\hat{\mathbf{v}}^H\hat{\mathbf{T}}\hat{\mathbf{v}}} \\ &= \frac{\hat{\mathbf{v}}^H\hat{\mathbf{T}}\mathbf{v}}{\hat{\mathbf{v}}^H\hat{\mathbf{T}}\hat{\mathbf{v}}}s_N(k) + \sum_{n=1}^{N-1}\frac{\hat{\mathbf{v}}^H\hat{\mathbf{T}}\mathbf{a}_n}{\hat{\mathbf{v}}^H\hat{\mathbf{T}}\hat{\mathbf{v}}}s_n(k) + \frac{\hat{\mathbf{v}}^H\hat{\mathbf{T}}}{\hat{\mathbf{v}}^H\hat{\mathbf{T}}\hat{\mathbf{v}}}\mathbf{u}(k) \\ &\triangleq \hat{\alpha}s_N(k) + \sum_{n=1}^{N-1}\hat{\beta}_n s_n(k) + \hat{\boldsymbol{\mu}}^H\mathbf{u}(k)\end{aligned} \tag{15}$$

where $\mathbf{a}_n$ is the n-th column of $\mathbf{D}$ and $\alpha, \beta_n, \boldsymbol{\mu}$ are defined by the equation above. The last equation indicates that the estimate of the signal waveform is corrupted by a multiplicative complex scalar, an additive residuals of all the co-channel signals, and an additive scalar function of the noise in all the antennas.

It is of interest to examine in detail each of these contributors to the error in the signal reconstruction. We have already seen that under *ideal conditions* we have

$$\begin{aligned}&\alpha = 1, \\ &\beta_n = 0, \quad n = 1, 2, \cdots, N-1;\end{aligned} \tag{16}$$

We can now define the Signal to Interference Ratio (SIRO) at the array output as,

$$\text{SIRO} \triangleq \frac{p_N \text{E}\{|\hat{\alpha}|^2\}}{\sum_{n=1}^{N-1} p_n \text{E}\{|\hat{\beta}_n|^2\}} \tag{17}$$

where p_n stands for the power of the n-th signal.

Note that $\hat{\beta}_n$ is random due to the random errors in estimating the steering vectors. In order to evaluate $\text{E}\{|\hat{\beta}_n|^2\}$ we examine the sensitivity of $\hat{\beta}_n$ to small errors in the estimation of the steering vectors.

A first order Taylor expansion of β_n is given by

$$\hat{\beta}_n \simeq \sum_{k=1}^{M}\frac{\partial\beta_n}{\partial g_k}\delta g_k + \sum_{p=1}^{M}\sum_{q=1}^{N}\frac{\partial\beta_n}{\partial\phi_{pq}}\delta\phi_{pq} \tag{18}$$

Using straightforward algebraic manipulations we obtain the following results,

$$\frac{\partial\beta_n}{\partial g_k} = -\frac{\mathbf{e}_k^T\tilde{\mathbf{C}}\mathcal{D}\mathbf{a}_n\mathbf{v}^H\mathbf{T}\mathbf{e}_k}{\mathbf{v}^H\mathbf{T}\mathbf{v}} \tag{19}$$

where $\tilde{\mathbf{C}}$ represents the matrix $\mathbf{C}$ with the last column omitted and $\mathbf{e}_k$ represents a column vector whose elements are zero except for the kth element which is one. We also get,

$$\frac{\partial\beta_n}{\partial\phi_{pq}} = \begin{cases}-j\dfrac{\mathbf{e}_p^T(\mathcal{D}\mathbf{a}_n\mathbf{v}^H\mathbf{T})^T\mathbf{e}_q\mathbf{e}_p^T\mathbf{D}\mathbf{e}_q}{\mathbf{v}^H\mathbf{T}\mathbf{v}} & \text{for } q < N \\ 0 & \text{for } q = N\end{cases} \tag{20}$$

Let us define

$$\mathbf{u} \triangleq [\cdots g_k\cdots, \cdots\phi_{pq}\cdots]^T \tag{21}$$

$$\mathbf{h}_n \triangleq [\cdots\frac{\partial\beta_n}{\partial g_k}\cdots, \cdots\frac{\partial\beta_n}{\partial\phi_{pq}}\cdots]^T \tag{22}$$

Using these definitions the denominator of (17) can be written as

$$\sum_{n=1}^{N-1} p_n\text{E}\{|\hat{\beta}_n|^2\} = \sum_{n=1}^{N-1} p_n\mathbf{h}_n^H\text{E}\{\delta\mathbf{u}\delta\mathbf{u}^T\}\mathbf{h}_n \tag{23}$$

In order to evaluate $\text{E}\{\delta\mathbf{u}\delta\mathbf{u}^T\}$ we use the Cramer Rao bound on estimating the steering vectors, which is derived in the Appendix of [10]. A close approximation for $\text{E}\{|\hat{\alpha}|^2\}$ in the numerator of (17) is 1, provided that a reasonable number of snapshots is used (i.e. $\text{E}\{|\hat{\alpha}|^2\} = 1 + O(1/N_s)$). We therefore use the approximation

$$\text{SIRO} \simeq \frac{p_N}{\sum_{n=1}^{N-1} p_n\mathbf{h}_n^H\text{CRBS}\mathbf{h}_n} \tag{24}$$

where CRBS stands for the Cramer Rao Bound on the estimation error of the Steering vectors.

In a similar way we define the output Signal to Noise Ratio (SNRO) as

$$\text{SNRO} \triangleq \frac{p_N\text{E}\{|\hat{\alpha}|^2\}}{\text{E}\{||\hat{\boldsymbol{\mu}}||^2\}\eta} \tag{25}$$

For large N_s we have

$$\text{SNRO} \simeq \frac{p_N(\mathbf{v}^H\mathbf{T}\mathbf{v})^2}{\eta(\mathbf{v}^H\mathbf{T}\mathbf{v})} = \frac{p_N}{\eta}\mathbf{v}^H\mathbf{T}\mathbf{v} \tag{26}$$

The last equation indicates that the SNRO is a function of $\mathbf{v}^H\mathbf{T}\mathbf{v}$ which may be very small if the desired and interfering signals are not well separated. In such a case, the cancellation of the interference causes a significant cancellation of the desired signal as well.

4. NUMERICAL EXAMPLES

In this section we present some numerical examples which show the quality of signal estimation which can be achieved using uncalibrated arrays. Each plot shows theoretical results by solid lines and experimental results by small 'x'. The experimental results are obtained by Monte Carlo computer simulations using the algorithm described in [1]. Each 'x' represent 300 experiments.

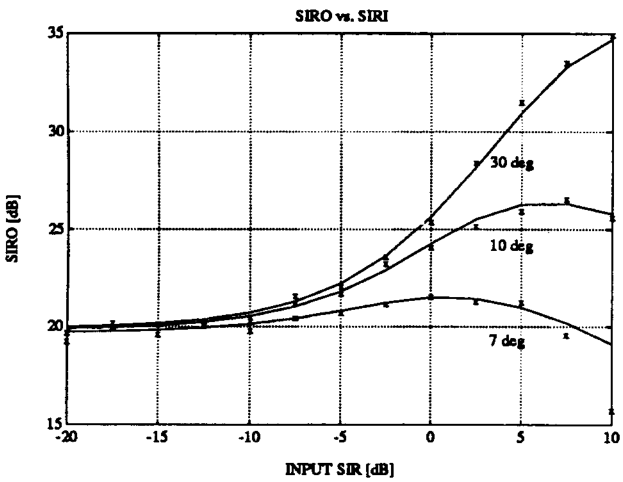

Figure 1: SIRO vs. SIRI. The interference intensity is decreased while the signal intensity is fixed.

Figure 1 shows the SIRO as a function of SIRI for different DOA separations. Here we kept the signal power fixed (input signal to noise ratio is 10 dB) and decreased the interference power. The array is a 5 element linear array with element spacing of half wavelength. We observe that for DOA separation of 7° the SIRO decreases even though the SIRI increases. The reason for this is the increasing difficulty in estimating precisely the steering vector associated with the interference. Since the desired signal is close to the interference, the interference suppression is reduced. This effect disappears as the signal separation increases.

5. CONCLUSIONS

In this work we examined the possibility of using uncalibrated arrays for separation and reconstruction of superimposed signals arriving from different directions. We used small error analysis and the CRB, to calculate the output signal to interference ratio and the output signal to noise ratio. The theoretical analysis was verified by Monte Carlo computer simulations, using an algorithm for steering vector estimation presented in [1]. The possibility of signal estimation and separation with uncalibrated arrays is of considerable practical importance. We are currently working on the development of computationally efficient algorithms for performing the estimation.

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