PERFORMANCE OF FH SS RADIO NETWORKS WITH INTERFERENCE MODELED AS A MIXTURE OF GAUSSIAN AND ALPHA-STABLE NOISE

J. Ilow *D. Hatzinakos* *A.N. Venetsanopoulos*

Department of Electrical and Computer Engineering, University of Toronto,
10 King's College Road, Toronto, Ont. M5S-1A4

ABSTRACT

This paper [1] considers the performance of FH SS radio networks in a Poisson field of interfering terminals using the same modulation and power. Assuming $\frac{1}{r^m}$ attenuation of signal strength over distance, the interference in the network is modeled as a mixture of Gaussian and circularly symmetric α-stable noise. The problem is relevant to mobile communication systems where the mobility of users requires random modeling of transmitter positions in the network.

In this paper, we derive formulas for probability distributions of the interference in FH SS radio networks. We generalize the results of Ref. [1], where attention was limited to Cauchy RV's, a special subclass of stable distributions, and where the effect of a background (Gaussian) noise was neglected. Based on the formulas derived, we calculate the probability of symbol error for radio links in environments varying from urban settings to office buildings. The results obtained allow the prediction of the performance of wireless systems under a wide range of conditions.

1. INTRODUCTION

For frequency-hopped spread-spectrum (FH SS) radio networks, it is of interest to determine the probability of symbol error for environments characterized by different attenuation of signal strength over distance [2], [3]. The interference at the receiver has two components; self-interference and external interference, such as thermal noise. The network self-interference depends on: 1) the positions of network terminals; and 2) the transmission characteristics of each terminal. However, terminal positions are usually unknown. To obtain the average performance of the network, it is often assumed that the terminal positions are randomly distributed on the plane. Assuming a $\frac{1}{r^{2m}}$ propagation power loss law and in considering a link in a radio network that is affected by a Poisson field [4] of interfering transmitters, it has been shown [1] that self-interference has a circularly symmetric α-stable (CS αS) distribution [5], [6], [7]; this distribution depends on propagation conditions. In [1], the author limits his attention to multivariate Cauchy random variables (RV's), a special subclass of stable distributions. Moreover, the effect of a background, Gaussian noise is neglected.

Most of the difficulty in analyzing systems with noise modeled as i.i.d., stable RV's arises because there are no closed-form expressions for the the probability density functions (pdf's) of stable distributions. The problem seems to be even more complicated when one models noise as a mixture of αS and Gaussian RV's.

In this paper, we consider the noise at the receiver input in FH SS radio networks as an additive combination of Gaussian and αS noises. We concentrate on circularly symmetric bivariate distributions. The objective of this paper is to derive the convergent series for the pdf of distributions under investigation and to calculate the probability of error (Pe) for FH with an on-off keying (OOK) and non-coherent reception. The results are given in terms of network parameters and a length R of the transmitter-receiver link.

The results obtained are new, and they give more insight into the analysis of the kind of systems described in this paper.

2. SYSTEM AND INTERFERENCE MODELS

We assume that the signal amplitude loss function with the distance r between a terminal and a receiver is given by

$$a(r) = \frac{1}{r^m}. \tag{1}$$

In free space, when radio frequency (RF) power radiates perfectly in a sphere from the antenna, the received power will decay in proportion to the square of the distance between the transmitter and receiver corresponding to a value of $m = 1$ in (1). In practice, m can vary from slightly less than 1 for hallways within buildings to larger than 3 for dense urban environments and office buildings [3].

We make the assumption that transmitting terminals are distributed on the plane in accordance with the following conditions:

1. The probability of k transmitters, which use the same frequency, being in a region $\mathbf{R}$ depends only on the area A of the region $\mathbf{R}$, and not on its shape nor its position on the plane. This probability is is given by:

$$P[k \text{ in } \mathbf{R}] = \frac{e^{-\lambda A}(\lambda A)^k}{k!}. \tag{2}$$

2. The numbers of terminals falling in non-overlapping regions are independent RV's.
3. The conditional distribution of the position of an ensemble point in $\mathbf{R}$ under the condition that it falls in this region is uniform, i.e., it has a distribution density equal to $1/A$.

Consequently, terminals using the same frequency form a Poisson process with the expected number of terminals per unit area given by λ. The conditions imposed on the process

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characterizing the positions of terminals are satisfied with great accuracy in many radio networks.
After being demodulated, the received signal is

$$z(t) = s(t) + \sum_i a(r_i)x_i(t) + n(t), \quad (3)$$

where $s(t)$ is the signal of interest, $n(t)$ is Gaussian noise, and the sum is the self-interference signal. The summation is taken over all the interfering terminals from a Poisson field [1]. After projecting the baseband signal onto the set of n basis functions, the received waveform is represented by an n dimensional vector:

$$\mathbf{Z} = \mathbf{S} + \sum a(r_i)\mathbf{X}_i + \mathbf{N} = \mathbf{S} + \mathbf{Y} + \mathbf{N}. \quad (4)$$

Because we consider the noncoherent detection and terminals use the same modulation scheme, it is reasonable to assume that the random vectors $\mathbf{X}_i$ are independent and identically distributed. Also, because all terminals transmit at the same power, the distribution of $\mathbf{X}_i$ is independent of r_i.
If the RV's $\mathbf{X}_i$ are i.i.d. and CS, it is shown in Appendix A that the characteristic function of the self-interference vector $\mathbf{Y}$ is stable, i.e.,

$$\phi_{\mathbf{Y}}(\mathbf{t}) = \exp(-\beta\lambda|\sum_{i=1}^{n} t_i^2|^{\frac{1}{2}\alpha}) = \exp(-\gamma|\mathbf{t}|^{\alpha}), \quad (5)$$

where $|\mathbf{t}|$ is the Euclidean (l_2) norm of the vector $\mathbf{t}$. The parameter β is given as

$$\beta = -\pi \int_0^{\infty} \frac{\Phi_0'(x)}{x^{\alpha}} dx, \quad (6)$$

where $\alpha = 2/m$; $\Phi_0(x) = \Phi_{\mathbf{X}}(|\mathbf{t}|)$ is a characteristic function of CS RV $\mathbf{X}_i$; and $'$ denotes differentiation. This result was first obtained in [1]. The proof presented in [1] is based on the influence function approach [6] for calculating the characteristic function of a multivariate $\mathbf{Y}$. Our proof exploits the series representation of univariate α-stable R.V.'s. It gives a more probabilistic interpretation of $\mathbf{Y}$ as opposed to the analytic one presented in [1]. Moreover, our proof gives new insight into the origin of stable distributions.
The characteristic exponent α in (5) controls the heaviness of the pdf tails ($0 < \alpha \leq 2$): a small positive value of α indicates severe impulsiveness, while a value of α close to 2 indicates a more Gaussian type of behavior [7]. Like a variance for Gaussian RVs, the dispersion γ is the scale parameter from $(0, \infty)$ and controls the spread around the origin.
We assume that the Gaussian vector $\mathbf{N}$ is circularly symmetric, i.e., it has a pdf given by $\mathcal{N}_n(\mathbf{0}, \sigma^2\mathbf{I})$, where $\mathbf{I}$ is the identity matrix. Also, the the Gaussian noise component $\mathbf{N}$ is independent of the self-interference noise component $\mathbf{Y}$. Therefore, the characteristic function of the total noise is:

$$\phi_{\mathbf{Y}+\mathbf{N}}(\mathbf{t}) = \psi(|\mathbf{t}|) = \exp(-\frac{\sigma^2}{2}|\mathbf{t}|^2 - \gamma|\mathbf{t}|^{\alpha}) \quad (7)$$

In the case of FH with OOK, where a sinusoidal tone is transmitted as the "on" symbol, the interference consists of sinusoidal tones ($n = 2$); thus $\mathbf{X}_i = [\cos(\Theta_i), \sin(\Theta_i)]$ where Θ_i is uniformly distributed in $[0, 2\pi]$. Moreover, $\Phi_0(x) = J_0(x)$ [10], where $J_\nu(\cdot)$ is a νth order Bessel function of the first kind. Because $J_0'(x) = -J_1(x)$, the formula 6.561.17 (p.708) from [9] can be used to calculate that for $0.5 < \alpha < 2$,

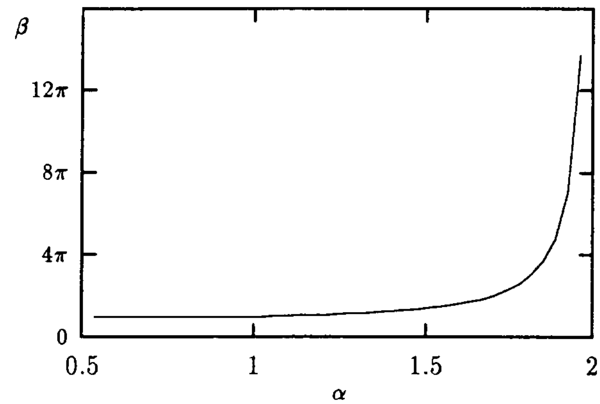

Figure 1. Dispersion as a function of α

$$\beta = \pi \frac{\Gamma(1-\alpha/2)}{2^{\alpha}\Gamma(1+\alpha/2)}. \quad (8)$$

In this case, the admissible range of the path loss exponent is $1 < m < 4$.
In Fig.1, $\beta = \gamma/\lambda$ is plotted as a function of α. For a wide range of α's, the dispersion is almost constant. As α decreases, the tails of the CS α-stable distribution become heavier and, if the dispersion is constant, the area of the tails will increase. This means that the channel will be more impulsive. Next, consider the range of α's in which a considerable increase in dispersion is observed with increasing α. As α decreases, the tails area does not necessarily increase at the same rate as in the previous range.

3. MIXTURE OF CS 2-D GAUSSIAN AND α-STABLE DISTRIBUTIONS

To evaluate the probability of symbol error for FH/OOK with an envelope detector, we first determine the distribution of the total interference $\mathbf{Y} + \mathbf{N}$.
If $\mathbf{Y}$ is independent of the Gaussian vector $\mathbf{N} \sim \mathcal{N}_n(\mathbf{0}, \sigma^2\mathbf{I})$, then [10] in the case of two-dimensional (2-D) distributions, the pdf of $\mathbf{Y} + \mathbf{N}$ is

$$p_{\mathbf{Y}+\mathbf{N}}(\mathbf{z}) = \frac{1}{2\pi\sigma^2} \int_0^{\infty} p_{|\mathbf{Y}|}(v) \exp[\frac{-1}{2\sigma^2}(v^2 + |\mathbf{z}|^2)] I_0(\frac{v|\mathbf{z}|}{\sigma^2}) dv, \quad (9)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind. To evaluate (9) when $\mathbf{Y}$ is CS αS, we use the power series representation for densities of $|\mathbf{Y}|$ [6]:

$$p_{|\mathbf{Y}|}(y) = \frac{1}{\alpha\gamma^{2/\alpha}} |y| \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\frac{2k+2}{\alpha})}{[\Gamma(k+1)]^2} (\frac{1}{2}\gamma^{\frac{-1}{\alpha}} |y|)^{2k}. \quad (10)$$

Because $I_0(x) = J_0(jx)$ [8], integrating term-by-term in (9) and based on a formula 8.406.3, p.961, in [9] for $\int_0^{\infty} J_0(at)\exp(-p^2t^2)t^{\mu-1}dt$, we arrive at the relation

$$p_{\mathbf{Y}+\mathbf{N}}(\mathbf{z}) = \frac{\gamma^{-2/\alpha}}{2\pi\alpha} \sum_{k=0}^{\infty} h_k L_k(-\frac{z^2}{2\sigma^2}), \quad (11)$$

where $h_k = (-1)^k(\frac{0.5\sigma^2}{\gamma^{2/\alpha}})^k\frac{\Gamma(\frac{2k+2}{\alpha})}{k!}$, and $L_k(\cdot)$ is a kth order Laguerre polynomial.

The application of formula (11) is limited because of the oscillating character of coefficients h_i. Using the inverse-Fourier approach, the pdf of 2-D CS RV $\mathbf{Y}+\mathbf{N}$ is also given by the Hankel transform [11]

$$p_{\mathbf{Y}+\mathbf{N}}(\mathbf{z}) = \frac{1}{2\pi}\int_0^\infty \psi(x)x\mathcal{J}_0(|\mathbf{z}|x)dx, \tag{12}$$

where $\psi(\cdot)$ is the generating characteristic function of $\mathbf{Y}+\mathbf{N}$ as in (7). In numerical calculations, (12) is more practical. Although we evaluated in this section the pdf of 2-D $\mathbf{Y}+\mathbf{N}$, the same approaches are applicable for higher dimensions.

4. PROBABILITY OF SYMBOL ERROR

For equiprobable binary signaling in the FH/OOK scheme, two hypotheses are considered. The first H_1 states that "1" is transmitted, and the alternative H_0 states that no signal is present. For maximum likelihood reception, the calculations are based on the joint pdf of 2-D RV $\mathbf{Z} = (Z1, Z2)$ as in (4). The receiver computes the following likelihood ratio:

$$\Lambda(\mathbf{z}) = \frac{p_{\mathbf{Z}|H_1}(\mathbf{z})}{p_{\mathbf{Z}|H_0}(\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} 1. \tag{13}$$

Under the hypothesis H_1, the RV's Z_1 and Z_2 will have the signal components $(\cos\Theta)/R^m$ and $(\sin\Theta)/R^m$, respectively, where Θ is the phase angle uniformly distributed on $[0, 2\pi]$ and R is the link distance. Thus,

$$\begin{aligned} H_0 &: p_{\mathbf{Z}|H_0}(\mathbf{z}) = p_{\mathbf{Y}+\mathbf{N}}(z1, z2) \\ H_1 &: p_{\mathbf{Z}|H_1}(\mathbf{z}) = \frac{1}{2\pi}\int_0^{2\pi} p_{\mathbf{Y}+\mathbf{N}}(z1 - \frac{\cos\Theta}{R^m}, z2 - \frac{\sin\Theta}{R^m})d\Theta \end{aligned} \tag{14}$$

After some manipulation, it can be shown that the likelihood ratio in (13) is a function of single variable $\rho = \sqrt{z_1^2 + z_2^2}$

$$\Lambda(\rho) = \frac{\int_0^{2\pi}\int_0^\infty \psi(\mathrm{x})\mathrm{x}\mathcal{J}_0(\mathrm{x}\sqrt{\rho^2 + \frac{1}{\mathrm{R}^{2m}} - \frac{2\rho\cos\Theta}{\mathrm{R}^m}})\mathrm{d}\mathrm{x}\mathrm{d}\Theta}{2\pi\int_0^\infty \psi(\mathrm{x})\mathrm{x}\mathcal{J}_0(\mathrm{x}\rho)\mathrm{d}\mathrm{x}} \tag{15}$$

For sub-optimum reception, we need the threshold ρ^* for which $\Lambda(\rho) = 1$. It is difficult to find a general formula for ρ^*, so we approximated it for different values of α corresponding to different environments (see Table 1). The results obtained depend on the expected density of terminals λ using the same frequency. We express our results in terms of the parameter N, which is defined after [1] as $N = \lambda\pi R^2$. The parameter N is equal to the expected number of interferers that are closer to the receiver than is the transmitter. We assume that the level of Gaussian noise relative to that of the α-stable noise is $0.5\sigma^2 = 0.1\gamma$. The average Pe is computed as

$$Pe = 0.5(P[\mathrm{error}|0] + P[\mathrm{error}|1]). \tag{16}$$

The conditional Pe given "0" can be calculated numerically using [11]

$$\mathrm{P}[\mathrm{error}|0] = \mathrm{P}(|\mathbf{Z}| > \rho^* \mid \mathrm{H}_0) = 1 - \int_0^\infty \psi(\mathrm{x})\mathrm{x}\mathcal{J}_1(\mathrm{x}\rho^*)\mathrm{dx} \tag{17}$$

The conditional Pe given "1" can be obtained from (14) based on (11) or using some form of approximation based on (12). Results for $N = 0.01$ and $N = 0.04$ with $R = 1$ are summarized in Table 1.

The results presented in Table 1 were obtained using Matlab. The number of terms used in calculating infinite series and integrals was adjusted as to achieve convergence in the last significant digit. This approach could result in small quantitative errors; however, it gives a general idea of the link-level performance of the radio network.

In general, the Pe deteriorates as the parameter N increases. This is expected because, as N increases, the self-interference becomes stronger. Moreover, in the cases examined, the performance of the link improves with a decrease in α because the dispersion of the noise is smaller with decreasing α. This effect counterbalances an increase in impulsiveness of noise with decreasing α. For optimum performance, the threshold ρ^* should increase with an increase of N and decrease with decreasing α.

5. CONCLUSION

We have considered a link in the FH SS radio network that is affected by a Poisson field of transmitting terminals. A mixture of Gaussian and α-stable noise was used to model the system interference. A new proof was given showing that the self-interference is indeed CS α-stable in the networks under consideration. The functional series representation was obtained for the pdf of the system interference. Based on the formulas derived, the probability of error was examined for radio links in different environments.

With the proliferation of personal communications systems, analytical results that describe the performance of wireless systems in realistic situations are becoming increasingly important. Therefore, we expect that the analysis presented will be useful in systems where non-Gaussian statistics are important.

A SELF-INTERFERENCE DISTRIBUTION

In the following, we give a new proof that the characteristic function of the self-interference RV $\mathbf{Y}$ is indeed α-stable. Without loss of generality, we assume that $\lambda = 1$. We rewrite $\mathbf{Y}$ as

$$\mathbf{Y} = \pi^{\frac{1}{\alpha}}\sum\frac{1}{(\pi r_i^2)^{\frac{1}{\alpha}}}\mathbf{X}_i. \tag{18}$$

For Poisson points distributed on the plane, πr_i^2 represents "occurrence" times, and for a homogeneous field, πr_i^2 is gamma distributed [4]. In each coordinate, the sum in (18) has the form

$$Y_{1/2} = \sum\Gamma_i^{-\frac{1}{\alpha}}X_{i,1/2}, \tag{19}$$

where $X_{i,1/2}$ is a symmetric univariate distribution. From Theorem 1.4.2 in [7], (p.23), a RV $Y_{1/2}$ is symmetric α-stable ($S\alpha S$) with the dispersion

$$\gamma = C_\alpha^{-1}E \mid X_i \mid^\alpha, \tag{20}$$

where $C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}$. Moreover, each summand $\Gamma_i^{-\frac{1}{\alpha}}X_{i,1/2}$ is univariate α-stable with the same dispersion. Because the bivariate RV $\mathbf{X}_i$ is CS, and because the coordinates of $\Gamma_i^{-\frac{1}{\alpha}}\mathbf{X}_i$ are α-stable with the same characteristic exponent α and dispersion, the sum in (18) is CS α-stable with the characteristic function

$$\phi_{\mathbf{Y}}(\mathbf{t}) = \exp(-\gamma|\mathbf{t}|^\alpha), \tag{21}$$

Table 1. Pe for different environments

Environment	Path Loss m	Ch. exp. α	Norm Dispersion $\beta = \gamma/\lambda$	$N = \lambda\pi R^2$ $[10^{-2}]$	Threshold ρ^*	$P[\text{error}\|0]$	$P[\text{error}\|1]$
Urban I	1.1	$\frac{20}{11}$	3.088π	1	0.66	0.041	0.053
				4	0.67	0.378	0.115
Urban II	1.5	$\frac{4}{3}$	1.177π	1	0.64	0.022	0.007
				4	0.65	0.083	0.035
Indoor I	2	1.0	π	1	0.63	0.029	0.005
				4	0.64	0.065	0.015
Indoor II	2.5	$\frac{4}{5}$	0.964π	1	0.63	0.032	0.004
				4	0.63	0.059	0.010
Residential	3	$\frac{2}{3}$	0.955π	1	0.63	0.033	0.003
				4	0.63	0.054	0.007

where γ is as in (20).
To complete the proof, we show that the dispersion in (20) is the same as the dispersion obtained in [1]

$$\beta = -\pi \int_0^\infty \frac{\Phi_0'(x)}{x^\alpha} dx, \tag{22}$$

where $\Phi_0(x) = \Phi_{\mathbf{X}}(|\mathbf{t}|)$ is a characteristic function of CS RV $\mathbf{X}_i$. First, we observe that the generating characteristic function $\Phi(\cdot)$ of the CS RV $\mathbf{Y}$ is the same as the characteristic function of $X_{i,1/2}$ [5]. Next, we manipulate $E \mid X_i \mid^\alpha$ into

$$\begin{aligned} E \mid X_i \mid^\alpha &= \int_{-\infty}^{\infty} \mid x \mid^\alpha p_X(x) dx \\ &= \int_{-\infty}^{\infty} \mid x \mid^{\alpha-1} \ \mathbf{sgn}(x) x p_X(x) dx, \end{aligned} \tag{23}$$

where $p_X(\cdot)$ is the pdf of $X_{i,1/2}$. To calculate $E \mid X_i \mid^\alpha$, we apply the Parseval Theorem to the last relation

$$E \mid X_i \mid^\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-2\pi \sin(\pi\alpha/2)\Gamma(\alpha)\mathbf{sgn}(x)}{\mid x \mid^\alpha} \frac{d\Phi_0(x)}{dx} dx. \tag{24}$$

The Fourier transform of $\mid x \mid^{\alpha-1}$ $\mathbf{sgn}(x)$ can be found in [9], (17.23.25,pg.1185). To show the equivalence of the dispersion in (20) and (22), we need

$$2C_\alpha \sin(\pi\alpha/2)\Gamma(\alpha) = \pi. \tag{25}$$

This can be verified using the following relations

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin(\pi z)}, \\ \Gamma(z+1) &= z\Gamma(z). \end{aligned} \tag{26}$$

REFERENCES

[1] E. Sousa, *Performance of a Spread Spectrum Packet Radio Network Link in a Poisson Field of Interferences*, IEEE Trans. I.T., Vol. 38, No.6, pp.1743–1754, Nov 1992.

[2] M.K. Simon, J.K. Omura, R.A. Scholtz and B.K Levitt, *Spread Spectrum Communications.* Rockville, MD: Computer Science Press, 1984.

[3] B.D. Woerner, J.H. Reed and T.S. Rappaport, *Simulation Issues for Future Wireless Modems*, IEEE Communications Magazine, pp.42–53, July 1994.

[4] D.L. Snyder and M.I. Miller, *Random Point Processes in Time and Space*, Springer-Verlag New York Inc., 1991.

[5] Fang K.-T., Kotz S and Ng Kai, *Symmetric Multivariate and Related Distributions*, Chapman and Hall, 1990.

[6] V. M. Zolotarev, *Integral Transformations of Distributions and Estimates of Parameters of Multidimensional Spherically Symmetric Stable Laws*, in Contributions To Probability, Ed.J.Gani, pp.283-305, Academic Press,1981.

[7] G. Samorodnitsky and M.S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman & Hall, New York, 1994.

[8] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, 1944.

[9] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Fifth Edition, Academic Press Inc., San Diago, 1994.

[10] J. Goldman, *Detection in the Presence of Spherically Symmetric Random Vectors*, IEEE Trans. I.T., Vol. 22, No.1, pp.52–59, Jan. 1976.

[11] R.D. Lord, *The Use of the Hankel Transform in Statistics*, Biometrica, Vol.41, pp.44–55, 1954.