FAST ESTIMATION OF THE PARAMETERS OF ALPHA-STABLE IMPULSIVE INTERFERENCE USING ASYMPTOTIC EXTREME VALUE THEORY

George A. Tsihrintzis

Department of Electrical Engineering
University of Virginia
Charlottesville, VA 22903-2442

Chrysostomos L. Nikias

Signal and Image Processing Institute
University of Southern California
Los Angeles, CA 90089

ABSTRACT

We address the problem of estimation of the parameters of the recently proposed symmetric, alpha-stable model for impulsive interference. We propose new estimators based on asymptotic extreme value theory, order statistics, and fractional lower-order moments, which can be computed fast and are, therefore, suitable for the design of real-time signal processing algorithms. The performance of the new estimators is evaluated theoretically and via Monte-Carlo simulation.

Key words: Impulsive Interference, Stable Distribution, Asymptotic Extreme Value Theory, Order Statistic, Fractional Lower-Order Moment.

1. INTRODUCTION

One physical process, which is not adequately described in terms of Gaussian models, is the process that generates "impulsive" interference bursts. Impulsive bursts occur in the form of short duration interferences, attaining large amplitudes with probability significantly higher than the probability predicted by a Gaussian probability density function (pdf). Many natural, as well as man-made, sources of impulsive inteference exist, including lightning in the atmosphere, switching transients in power lines and car ignitions, accidental hits in telephone lines, and ice cracking in the arctic region [1, 2]. Impulsive interference can cause significant degradation of the performance of conventional (Gaussian) communication, radar, and sonar systems and nonlinear signal processing algorithms are needed to filter it out [3, 4, 5, 6, 7].

Several models have been proposed for the description of impulsive interferences, including parametric statistical-physical models such as the Middleton model [3] and empirical models such as mixtures of Gaussian and non-Gaussian or other heavy-tailed distributions (e.g., [8, 9]) Very recently, a new statistical-physical model was proposed for impulsive interference [2, chapter 9]. In particular, it was theoretically shown in [2, chapter 9] that, under general assumptions, a broad class of impulsive noise has a *symmetric, alpha-stable* (SαS) first-order distribution [10, 2]. The new model provided a significant reduction in complexity when compared to the Middleton model; however, tests on a large variety of real data demonstrated no loss of accuracy relatively to the Middleton model [2, chapter 9].

The problem of estimation of the parameters of a SαS model has been addressed in the literature, mainly within the framework of Modern Statistics, and a number of approaches have been proposed to it (see [10, 2] and references therein). However, major difficulties are encountered when the classical estimation methods of Statistics are applied to this particular problem. The main source of these difficulties is the lack of closed-form expressions for the general SαS pdf. In this paper, we propose alternative estimators for the parameters of SαS distributions from observation of independent realizations of it. The new estimators are based on the asymptotic distributions of the extremes (maxima and minima) of collections of random variables, on order statistics, as well as on certain relations between fractional lower-order moments and the parameters of the distribution. These estimators are shown to maintain acceptable performance, while, at the same time, are simple enough to be computable in real time. These two properties of the proposed estimators render them very useful for the design of algorithms for statistical signal processing applications.

More specifically, the paper is organized as follows: In Section 2, we present theorems regarding the asymptotic distribution of the extreme order statistics of collections of independent, identically distributed (iid) SαS random variables with particular emphasis on the aspects of the theory that will be used in subsequent sections. In Section 3, we state the problem of estimation of the parameters of an iid sequence of stable random variables, which is the main concern of this paper, and propose estimators which can be computed very fast and are, therefore, suitable for real-time signal processing. The performance of these estimators is evaluated theoretically and via Monte-Carlo simulation. The paper is summarized and concluded in Section 4, in which we also propose avenues for future research.

This work was supported by the Office of Naval Research under contract N00014-92-J-1034.

2. ASYMPTOTIC DISTRIBUTION OF EXTREMES OF SαS RANDOM VARIABLES

Asymptotic Extreme Value Theory (AEVT) is the field of statistical analysis studying the distributions of extreme order statistics (maxima and minima) of collections of random variables. As such, it is very important in many engineering disciplines in which the laws of interest are governed by extremes. In the fields of communication theory and signal processing in particular, AEVT has found application in the estimation via extrapolation of very small probabilities involved in the assessment of the performance of communication devices and signal processing algorithms [11]. Two theorems are given here regarding the asymptotic distributions of the extremes of collections of iid SαS random variables. A more complete presentation of the field of AEVT and further applications can be found in the statistical literature [12, 11].

2.1. Symmetric, alpha-stable distributions

We can define the symmetric, alpha-stable (SαS) pdf $f(\cdot)$ via the inverse Fourier transform integral:

$$f_\alpha(\gamma,\delta;\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\delta\omega - \gamma|\omega|^\alpha)e^{-i\omega\xi}\,d\omega, \qquad (1)$$

where the parameters α (characteristic exponent), γ (dispersion), and δ (location parameter) relate to the heaviness of the tails, the spread, and the point of symmetry of the pdf, respectively [10, 2].

2.2. Frechet asymptotic distribution of extreme order statistics

Let $X_1, X_2, \ldots, X_K$ be a collection of independent realizations of a SαS random variable with pdf (*parent pdf*) $f_\alpha(\cdot)$ and cumulative distribution function (cdf) $F_\alpha(\cdot)$. Let $\overline{X}$ and $\underline{X}$ denote the maximum and the minimum in the sequence. We will refer to $\overline{X}$ and $\underline{X}$ as the *extreme order statistics* of the collection. Let us also define the *transformed* extreme order statistics $\overline{x} = \log\overline{X}$ and $\underline{x} = -\log(-\underline{X})$ [1].

Theorem 1 *There exists a sequence* $\{b_K\}$, $K = 1, 2, 3, \ldots,$ *such that* $b_K > 0$ *for all* K *and, as* $K \to \infty$,

$$b_K f_{M:K}(b_K\xi) \;\to\; \overline{f}_\alpha(\xi) \qquad (2)$$
$$b_K f_{m:K}(b_K\xi) \;\to\; \underline{f}_\alpha(\xi) \qquad (3)$$

where $f_{M:K}(\cdot)$ *and* $f_{m:K}(\cdot)$ *are the exact distributions of the maximum- and minimum-order statistics, respectively, of a sequence of length* K *[11, chapter 2] and*

$$\overline{f}_\alpha(\xi) = \begin{cases} \xi^{-\alpha-1}\alpha\exp(-\xi^{-\alpha}) & \text{if } \xi \geq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (4)$$

$$\underline{f}_\alpha(\xi) = \begin{cases} (-\xi)^{-\alpha-1}\alpha\exp(-[-\xi]^{-\alpha}) & \text{if } \xi \leq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (5)$$

[1]It can be shown that, as $K \to \infty$, $\overline{X} > 0$ and $\underline{X} < 0$ with probability one.

The distributions $\overline{f}_\alpha(\cdot)$ and $\underline{f}_\alpha(\cdot)$ in Eqs.(3) and (4) are the *Frechet* distributions for maxima and minima, respectively.

For the transformed extreme order statistics, we have the following:

Theorem 2 *Given the sequence* $\{b_K\}$, $K = 1, 2, 3, \ldots,$ *of Theorem 1, the asymptotic distribution (Gumbel distribution) of the transformed extremes, as* $K \to \infty$, *is*

$$f^G_{M,m}(\overline{x}_l, \underline{x}_l) = \frac{\exp(-\frac{\overline{x}_l - \overline{\lambda}_{G,K}}{b_G})\exp[-\exp(-\frac{\overline{x}_l - \overline{\lambda}_{G,K}}{b_G})]}{b_G} \cdot \frac{\exp(-\frac{\underline{\lambda}_{G,K} - \underline{x}_l}{b_G})\exp[-\exp(-\frac{\underline{\lambda}_{G,K} - \underline{x}_l}{b_G})]}{b_G}, \qquad (6)$$

where $b_G = 1/\alpha$, $\overline{\lambda}_{G,K} = \log(b_K)$, *and* $\underline{\lambda}_{G,K} = -\log(b_K)$.

The proof of Theorems 1 and 2 can be found in [13]. These results allow us to approximate the exact distribution of the extreme order statistics, which depends strongly on knowledge of the exact parent pdf, with asymptotic distributions, valid for long sequences (large K) and dependent only on the tail behavior of the parent pdf.

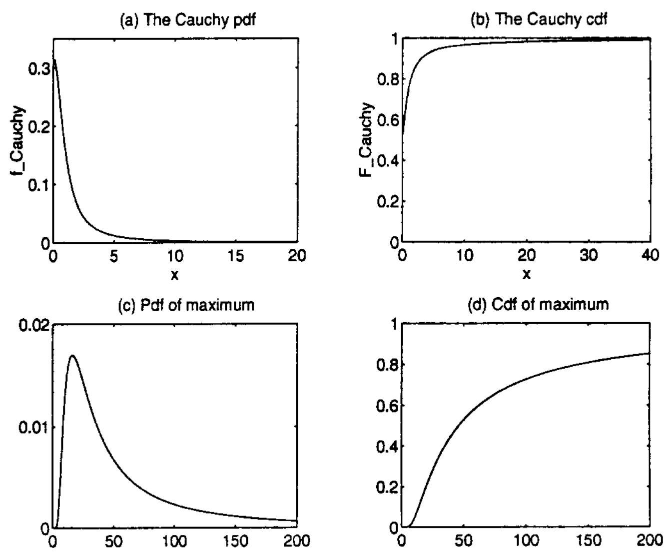

Figure 1: Illustration of the convergence to the asymptotic extreme distribution for a Cauchy (i.e., S($\alpha = 1$)S) parent pdf: (a) The Cauchy pdf; (b) The Cauchy cdf; (c) Exact (solid line) and asymptotic (dotted line) pdf of the maximum of a collection of $K = 100$ iid Cauchy random variables; (d) Exact (solid line) and asymptotic (dotted line) cdf of the maximum of a collection of $K = 100$ iid Cauchy random variables.

The convergence of the true extreme distribution to the asymptotic one is illustrated in Fig. 1 for a Cauchy pdf.

In particular, Fig. 1a shows the Cauchy pdf of zero location parameter and unit dispersion, i.e, the pdf $f_1(1,0;\xi) = (1/\pi)[1/(1+\xi^2)]$. Fig. 2a shows the corresponding Cauchy cdf, i.e., $F_1(1,0,\xi) = 1/2 + \arctan(x)/\pi$. The exact (solid line, $f_{M,K=100}(\xi) = 100F_1^99(1,0;\xi)f_1(1,0;\xi)$) and the asymptotic (dashed line, $(1/b_{K=100}\overline{f}_1(\xi/b_{K=100}))$ pdf of the maximum of a collection of $K = 100$ iid realizations of the Cauchy random variable are shown in Fig. 1c. Finally, Fig. 1d shows the corresponding extreme cdfs. From Figs. 1c and 1d, the high accuracy of the asymptotic approximation becomes clear, since it is almost impossible to tell the exact from the asymptotic distribution.

3. ESTIMATION OF THE PARAMETERS OF ALPHA-STABLE IMPULSIVE INTERFERENCE

3.1. Problem formulation

Let $X_1, X_2, \ldots, X_N$ be observed independent realizations of a SαS random variable X of unknown characteristic exponent α, location parameter δ, and dispersion γ. We attempt to estimate the exact parameters of the SαS distribution of X from the observed realizations.

3.2. Proposed algorithm for estimation of the characteristic exponent α

Consider a segmentation of the data into L nonoverlapping segments, each of length $K = N/L$:

$$\{X_1, X_2, \ldots, X_N\} = \{\mathbf{X}(1), \mathbf{X}(2), \ldots, \mathbf{X}(L)\}, \quad (7)$$

where $\mathbf{X}(l) = \{X_{(l-1)N/L+1}, X_{(l-1)N/L+2}, \ldots, X_{lN/L}\}$, $l = 1, 2, \ldots, L$.

Let $\overline{X}_l$ and $\underline{X}_l$ be the maximum and the minimum of the data segment $\mathbf{X}(l)$. We then define

$$\overline{x}_l = \log \overline{X}_l \quad (8)$$

$$\underline{x}_l = -\log(-\underline{X}_l) \quad (9)$$

and the corresponding standard deviations

$$\overline{s} = \sqrt{\frac{1}{L-1}\sum_{l=1}^{L}(\overline{x}_l - \overline{x})^2}; \qquad \overline{x} = \frac{1}{L}\sum_{l=1}^{L}\overline{x}_l \quad (10)$$

$$\underline{s} = \sqrt{\frac{1}{L-1}\sum_{l=1}^{L}(\underline{x}_l - \underline{x})^2}; \qquad \underline{x} = \frac{1}{L}\sum_{l=1}^{L}\underline{x}_l. \quad (11)$$

With these definitions in mind, the estimate for the characteristic exponent α of the SαS pdf takes the form

$$\hat{\alpha} = \frac{\pi}{2\sqrt{6}}\left(\frac{1}{\overline{s}} + \frac{1}{\underline{s}}\right) \quad (12)$$

Theorem 1 *The estimator $\hat{\alpha}$ of the characteristic exponent α of a SαS distribution is consistent and asymptotically normal with mean equal to the true exponent α and variance $\frac{9}{2L\pi^4}(\mu_{4,G} - \frac{L-3}{L-1}\frac{\pi^4}{36\alpha^4})\alpha^6$, as $N \to \infty$ and $L \to \infty$ such that $N/L \to \infty$. In the expression for the asymptotic variance of the estimator, $\mu_{4,G}$ is on the order of $\frac{1}{\alpha^4}$; therefore, the asymptotic variance of the estimator is on the order of α^2.*

3.3. Proposed algorithm for estimation of the location parameter δ

For the estimation of the location parameter δ of a SαS pdf, we propose the use of the sample median of the observations, i.e.

$$\hat{\delta} = \text{median}\ \{X_1, X_2, \ldots, X_N\}, \quad (13)$$

where the sample median is defined as follows: If the sample consists of an odd number N of observations, the median is defined as the center order statistic. If the sample consists of an even number N of observations, the median is defined as the average of the two center statistics. The sample median forms the maximum likelihood estimate of the location parameter of a Laplace (double exponential) distribution and, therefore, enjoys all the properties of maximum likelihood estimators in that case. Its performance as an estimator for the location parameter δ of a SαS pdf can be expected to be very robust. In fact, we have shown [13] that this estimator performs very closely to the maximum likelihood estimator for the case of a SαS pdf.

Theorem 2 *The estimator $\hat{\delta}$ of the location parameter δ of a SαS distribution is consistent and asymptotically normal with mean equal to the true parameter δ and variance $(\frac{\pi\alpha\gamma^{1/\alpha}}{2\Gamma(1/\alpha)})^2\frac{1}{N}$, as $N \to \infty$.*

3.4. Proposed algorithm for estimation of the dispersion γ

For the estimation of the dispersion γ of a SαS pdf, we propose the following estimator which is based on the theory of fractional lower order moments of the pdf:

$$\hat{\gamma} = \left[\frac{\frac{1}{N}\sum_{k=1}^{N}|X_k - \hat{\delta}|^p}{C(p,\hat{\alpha})}\right]^{\frac{\hat{\alpha}}{p}}, \quad (14)$$

where $C(p,\hat{\alpha})$ has been defined as

$$C(p,\hat{\alpha}) = \frac{1}{\cos(\frac{\pi}{2}p)}\frac{\Gamma(1-p/\hat{\alpha})}{\Gamma(1-p)} \quad (15)$$

and the choice of the order p ($0 < p < \frac{\hat{\alpha}}{2}$) of the fractional moment is arbitrary.

Theorem 3 *The estimator $\hat{\gamma}$ of the dispersion of a SαS distribution is consistent and asymptotically normal with mean equal to the true dispersion γ and variance $\frac{1}{N}(m_{2p} - m_p^2)\{\frac{\alpha}{p}\cdot\frac{(\frac{m_p}{C(p,\alpha)})^{\frac{\alpha}{p}}}{m_p}\}^2$, as $N \to \infty$ and $L \to \infty$ with $K = N/L \to \infty$. With m_p and m_{2p}, we have denoted the pdf moments of fractional orders p and $2p$, respectively.*

3.5. Monte-Carlo evaluation of the proposed algorithms

We tested the proposed algorithms via 1,000 Monte-Carlo runs in which we chose $N = 5,000$, $L = 50$ (i.e., $K = 100$), and $p = \frac{\hat{\alpha}}{4}$ for the cases $\alpha = 0.1$, 0.5, 1, and 1.5, $\gamma = 1$, and $\delta = 1$. The following tables summarize our findings for $\gamma = 1$ and $\gamma = 10$, respectively, showing the mean and the standard deviation (in brackets) of our proposed estimators.

Table 1: $\gamma = 1$				
	α			
	0.1	0.5	1.0	1.5
$\hat{\alpha}$	0.1031 (0.0107)	0.5167 (0.0538)	1.0351 (0.1072)	1.6107 (0.1725)
$\hat{\delta}$	1.000 (6.94×10^{-7})	0.9994 (0.0110)	0.9998 (0.0213)	0.9990 (0.0237)
$\hat{\gamma}$	1.0421 (0.1209)	1.0468 (0.1201)	1.0480 (0.1200)	1.0966 (0.1429)

Table 2: $\gamma = 10$				
	α			
	0.1	0.5	1.0	1.5
$\hat{\alpha}$	0.1030 (0.0109)	0.5116 (0.0532)	1.0386 (0.1066)	1.6140 (0.1760)
$\hat{\delta}$	58.6915 (8.4604×10^{8})	0.9979 (1.1096)	1.0106 (0.2129)	1.0025 (0.1138)
$\hat{\gamma}$	11.8845 (4.8725)	11.5582 (4.4013)	12.2093 (5.5068)	14.0672 (6.3089)

From Tables 1 and 2, we immediately draw the following conclusions:

1. The estimator $\hat{\alpha}$ of the characteristic exponent α becomes less accurate in terms of both bias and error variance as α gets closer to the Gaussian value $\alpha = 2$. Its performance, however, is independent of either the location parameter δ or the dispersion γ of the interference.
2. The estimator $\hat{\delta}$ of the location parameter δ is very efficient; however its performance decreases with decreasing α and increasing γ (i.e., with more impulsive interference).
3. The estimator $\hat{\gamma}$ of the dispersion γ requires either knowledge or prior estimation of both the characteristic exponent and the location parameter of the interference.

4. SUMMARY, CONCLUSIONS, AND POSSIBLE FUTURE WORK

In this paper, we presented algorithms for estimation of the parameters of symmetric, alpha-stable impulsive interference from independent observations. The algorithms employed several results from asymptotic extreme value theory, order statistics, and fractional lower-order moments and were analyzed both theoretically and via computer simulation. In the future, it seems interesting to extend the algorithms to the case of estimation of the parameters of arbitrary ARMA models from observation of their output only and to perform an extensive study of the properties and the performance of those and other related algorithms. This research has been initiated and its results will be announced shortly.

5. REFERENCES

[1] E. J. Wegman, S. G. Schwartz, and J. B. Thomas, eds., *Topics in Non-Gaussian Signal Processing.* New York: Academic Press, 1989.

[2] C. L. Nikias and M. Shao, *Signal Processing with Alpha-Stable Distributions and Applications.* New York, NY: John Wiley & Sons, Inc., 1995.

[3] D. Middleton, "Statistical-physical models of electromagnetic interference," *IEEE Trans. Electromagnetic Compatibility*, vol. EMC-19, no. 3, pp. 106–127, 1977.

[4] G. A. Tsihrintzis and C. L. Nikias, "Performance of optimum and suboptimum receivers in the presence of impulsive noise modeled as an α–stable process," *IEEE Trans. Comm.*, April 1995.

[5] G. A. Tsihrintzis and C. L. Nikias, "On the detection of stochastic impulsive transients over background noise," *Signal Processing*, February 1995.

[6] P. Tsakalides and C. L. Nikias, "A new robust array processing method based on the cauchy beamformer," in *Proc. 1994 Adaptive Sensor Array Processing Workshop*, (MIT Lincoln Lab, Lexington, MA), March, 1994.

[7] P. Tsakalides and C. L. Nikias, "Subspace-based direction finding in alpha-stable noise environments," in *Proc. ICASSP 1995*, (Detroit, MI), May 8-12 1995.

[8] P. Mertz, "Model of impulsive noise for data transmission," *IRE Trans. Comm. Systems*, vol. CS-9, pp. 130–137, 1961.

[9] M. Bouvet and S. C. Schwartz, "Comparison of adaptive and robust receivers for signal detection in ambient underwater noise," *IEEE Trans. Acoust., Speech, and Signal Process.*, vol. ASSP-37, pp. 621–626, 1989.

[10] M. Shao and C. L. Nikias, "Signal processing with fractional lower-order moments: Stable processes and their applications," *Proc. IEEE*, vol. 81, pp. 986–1010, 1993.

[11] E. Castillo, *Extreme Value Theory in Engineering.* San Diego, CA: Academic Press, 1988.

[12] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics.* Malabar, FL: Krieger, 1987.

[13] G. A. Tsihrintzis and C. L. Nikias, "Fast estimation of the parameters of alpha-stable impulsive interference," *IEEE Trans. Signal Processing*, p. 27, 1994. (submitted for publication; also available as USC-SIPI Technical Report # 270, September 1994).