ESTIMATION OF PHASE FOR NOISY LINEAR PHASE SIGNALS

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ABSTRACT

A procedure for estimating the parameters associated with a linear phase signal is developed. When the data being modelled is composed of a linear phase signal corrupted by additive Gaussian noise the approach taken results in maximum-likelihood estimates of the linear phase parameters. These estimates are useful for detecting and estimating the presence of symmetry in both one and two dimensions. The effectiveness of the estimates is tested on both synthetic and real images.

1. INTRODUCTION

The concept of symmetry plays an important role in digital signal processing. A generally complex-valued one-dimensional (1-D) signal is said to be symmetric about the point n_o if its elements satisfy the conjugate symmetry property $x(n-n_o) = \bar{x}(n_o-n)$ for all integer values of n. (The overbar symbol here being employed denotes complex-conjugation.) It is well-known that a symmetric signal has a linear-phase Fourier transform, as the phase spectrum of the transform is specified by $e^{-j\omega n_o}$. With these facts in mind, it is appropriate to consider the problem of detecting and estimating symmetry in data by examining the linearity of phase in the transform domain.

Formally, a sequence of complex numbers $s(n)$ is said to be *linear phase* if for appropriate real numbers r_n and angles θ, ϕ, we have

$$s(n) = r_n e^{j(\theta n + \phi)}. \qquad (1)$$

The parameters r_n are called the *amplitudes*, and they may be either positive, negative, or zero. The parameter θ is the *incremental phase*, and it measures the rate at which phase changes from one index to the next; the parameter ϕ is the *initial phase*, or equivalently the phase at $n = 0$. The primary aim of this paper is to develop methods for optimally estimating the parameters of the linear phase signal in (1) from noisy measurements. Suppose that in general there are available M observations x_m of the signal, with $1 \leq m \leq M$, where each x_m is of length N_m and moreover

$$x_m(n) = r_{mn} e^{j(\theta n + \phi_m)} + w_m(n); \quad 1 \leq n \leq N_m. \qquad (2)$$

The complex-valued term w_m is intended to represent either measurement noise, a portion of the signal that is not linear phase, or a combination of both. Without loss of generality the measurement interval for each observation is assumed to commence at index $n = 1$. The quality of estimates for the signal amplitudes $\{r_{mn}\}$, the initial phases $\{\phi_m\}$, and the incremental phase θ shall be measured by the normalized sum of squared errors criterion

$$\frac{\sum_{m=1}^{M} \sum_{n=1}^{N_m} |x_m(n) - r_{mn} e^{j(\theta n + \phi_m)}|^2}{\sum_{m=1}^{M} \sum_{n=1}^{N_m} |x_m(n)|^2} \qquad (3)$$

It is desired to select $\{r_{mn}\}$, $\{\phi_m\}$, and θ to minimize this criterion.

The parameter estimation literature contains results on special cases of the problem considered here. In particular, when all the amplitude terms r_{mn} in (2) are known to be equal to a constant A, the problem of estimating parameters to minimize (3) is the standard discrete-time frequency estimation problem that has been the subject of much research [1]. The principal new contributions of this paper are the following: (1) to determine the optimal estimates in the variable amplitude case; (2) to provide applications to symmetry axis estimation in image processing.

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2. OPTIMAL ESTIMATES

Noting that the generally suboptimal selection $r_{mn} \equiv 0$ causes (3) to equal one, we conclude that the minimum value assumed by this normalized criterion must always be contained the interval $[0,1]$. A criterion minimum value close to zero (one) is indicative of a good (poor) linear phase signal fit to the empirical data. It is to be further noted that when the additive term $\{w_m(n)\}$ corresponds to uncorrelated samples of a zero mean Gaussian random variable (i.e., white noise), then minimization of (3) leads to *maximum-likelihood estimates* of the linear phase parameters. Furthermore, if these samples of a Gaussian random variable are correlated (i.e., colored noise) then the approach now to be taken can be straightforwardly modified to obtain the desired maximum-likelihood estimates.

Minimization of (3) with regard to the estimates is carried out by equating to zero the partial derivatives with respect to the estimates. The resulting set of equations is straightforward to solve; we omit the details and simply state the results in the following theorem.

Theorem 1 *Let the measured data be specified by the multiple measurements model (2), in which the $\{w_m(n)\}$ terms represent uncorrelated samples of a zero mean Gaussian random variable. It then follows that the maximum likelihood estimates of the parameters θ, $\{\phi_m\}$, and $\{r_{mn}\}$ are given by minimizing the squared error criterion (3). Specifically, the optimum selection of the parameter θ is given by*

$$\theta^o = \arg\left\{\max_{\theta\in[0,\pi)}\left|\sum_{m=1}^{M}\sum_{n=1}^{N_m} x_m(n)^2 e^{j2\theta n}\right|\right\}, \quad (4)$$

and the optimum choices of the initial phase angle parameters are specified by

$$\phi_m^o = \frac{1}{2}\text{angle}\left\{\sum_{m=1}^{M}\sum_{n=1}^{N_m} x_m(n)^2 e^{j2\theta^o n}\right\}, \quad (5)$$

with the optimum amplitude parameters then being computed according to relationship

$$r_{mn}^o = \frac{x_m(n)e^{-j(\theta^o n+\phi_m^o)} + \bar{x}_m(n)e^{j(\theta^o n+\phi_m^o)}}{2}. \quad (6)$$

It is noted that the estimates of θ^o and ϕ_m^o may be implemented efficiently with the aid of the Fast Fourier Transform algorithm.

Although we do not provide the proof here, it may also be shown that these estimates are symmetrically distributed around their true values, and that the Cramér-Rao lower bounds on their variances are as follows (with σ_w^2 denoting the variance of the complex Gaussian noise):

$$\sigma^2[\theta^o] \geq \frac{\sigma_w^2}{2\sum_{m=1}^{M}\sum_{n=1}^{N_m} n^2 r_{mn}^2}; \quad \sigma^2[\phi_m^o] \geq \frac{\sigma_w^2}{2\sum_{n=1}^{N_m} r_{mn}^2}; \quad (7)$$

$$\sigma^2[r_{mn}^o] \geq \frac{\sigma_w^2}{2}. \quad (8)$$

Similar results have been obtained for the discrete-time frequency estimation problem [2]. However, the presence of variable amplitude terms r_{mn} makes the linear phase problem addressed here significantly different.

3. APPLICATIONS TO IMAGE PROCESSING

The estimation theory that is developed in the previous section, ostensibly for 1-D data, is also useful for analyzing symmetry in two-dimensions (2-D). This has applications in some image understanding problems—for example, automatic recognition of vehicle rears in an image [3]—where estimating symmetry parameters is important. While there are several types of symmetries in 2-D, we focus here on reflection symmetry about an axis in the plane, and we develop the optimal least-squares estimate of the reflection axis' inclination.

3.1. A useful decomposition

The transformation of any symmetric signal in the time domain to a corresponding linear phase signal in the frequency domain is one of the fundamentally important properties of the Fourier transform. It is appropriate therefore to seek an orthogonal decomposition for images that transforms reflection symmetry about an axis in the plane to a corresponding set of linear phase coefficients. One such decomposition is sketched below; a more detailed treatment is available elsewhere [4].

For any integer n, the n-th order Bessel function of the first kind, denoted J_n, is given by the integral

$$J_n(r) = \frac{1}{\pi}\int_0^{\pi}\cos(n\phi - r\sin(\phi))d\phi; \quad r \geq 0. \quad (9)$$

We use only the functions J_0 and J_1 in what follows. For $k \geq 0$, let α_k denote the nonnegative roots of the equation $rJ_1(r) = 0$, arranged in increasing order. These roots play the role of "frequencies" in the decomposition that is described below.

Theorem 2 *The following functions are orthonormal with respect to the polar measure $r\,dr\,d\theta$ on the disk $\{(r,\theta) : 0 \le r \le R,\ 0 \le \theta < 2\pi\}$:*

$$D_{n,k}(r,\theta) = \frac{1}{\sqrt{\pi}R|J_0(\alpha_k)|} J_0(\frac{\alpha_k}{R}r)e^{jn\theta}. \qquad (10)$$

Moreover, any square-integrable function f on the disk may be expressed in the following series:

$$f(r,\theta) = \sum_{k=0}^{\infty}\sum_{n=-\infty}^{+\infty} F(n,k)D_{n,k}(r,\theta), \qquad (11)$$

with the coefficients being determined by the integrals

$$F(n,k) = \int_0^{2\pi}\int_0^R f(r,\theta)\overline{D_{n,k}}(r,\theta)\, r\, dr\, d\theta. \qquad (12)$$

The series coefficients $F(n,k)$ are called the *Fourier-Dini coefficients* of the function f.

3.2. Properties of the Fourier-Dini coefficients

Let μ be the center of mass of an image f. It is easily shown that any reflection symmetry axis of f must pass through μ. Furthermore, let F_μ denote the Fourier-Dini coefficients of f, computed in a disk of radius R centered at μ. If the inclination of the axis of symmetry is θ_0 (measured counterclockwise from the horizontal), then we have the following identity:

$$f_\mu(r,\theta-\theta_0) = f_\mu(r,\theta_0-\theta). \qquad (13)$$

From (12) it follows that the corresponding Fourier-Dini coefficients must satisfy the identity

$$F_\mu(n,k)e^{-jn\theta_0} = \overline{F_\mu}(n,k)e^{jn\theta_0}. \qquad (14)$$

This in turn is satisfied if and only if the coefficients are linear in phase, i.e., for appropriate real numbers $R(n,k)$, we have

$$F_\mu(n,k) = R(n,k)e^{jn\theta_0}. \qquad (15)$$

In practice, it is more appropriate to model the observed coefficients of a reflectionally-symmetric image as follows:

$$F_\mu(n,k) = R(n,k)e^{jn\theta_0} + W(n,k) \qquad (16)$$

Here the complex-valued terms $W(n,k)$ represent either errors in digital computation of (12), errors in computing the center of mass μ, or the contribution from a portion of the image that is not perfectly symmetric. With such data, the optimal least-squares estimate of θ_0 is obtained by using the estimator in Theorem 1. It is noted that the initial phase parameters ϕ considered there are all zero in this particular application, although they may be otherwise in different applications.

Example 1 *Two sets of experiments were conducted to test the performance of the proposed symmetry axis estimation algorithm. The first set was performed on synthethic noisy images, and they investigated the effects of the following on estimator accuracy: digital computation of (12), the estimates in Theorem 1 being computed using relatively few Fourier-Dini coefficients, the signal-to-noise ratio, and the orientation of the symmetry axis. The second set of experiments investigated the effectiveness of the proposed estimators on various real images. All computations were done using the built-in Bessel function routines in MATLAB © on a standard 486-based PC. The results are now described.*

The following function is reflection symmetric about the y axis:

$$g(x,y) = yx^2e^{-(x^2+y^2)/2\sigma^2}. \qquad (17)$$

Synthetic images were generated by plotting various rotations of g on a 201×201 grid centered at the origin, with white Gaussian noise $n(x,y)$ added. The center of mass μ was estimated on each noisy image $f = g + n$ as follows:

$$\mu = \sum_x\sum_y \begin{bmatrix} x \\ y \end{bmatrix} f(x,y), \qquad (18)$$

the summation being carried out over the entire domain of the image. Obviously this estimate of μ is sensitive to the noise in f, but its reliability did not pose a problem in our experiments, even in high levels of noise. With $R = 50$ pixels, the Fourier-Dini coefficients $F_\mu(n,k)$ were computed for $1 \le n \le N$, and $1 \le k \le M$, where both $N = M = 2$ and $N = M = 3$ were used. The rationale for choosing these orders was to determine the effectiveness of the estimation procedure with relatively few coefficients. The inclination of the symmetry axis was then estimated from the coefficients by using the methods of Theorem 1. The entire estimation procedure, including calculation of the coefficients F_μ, took less than 5 seconds to compute for each image. Figure 1 shows a typical image, having 5 dB SNR, with the estimated axis superimposed. Overall, good performance was observed on these synthetic images.

The second set of experiments was conducted on various real images of symmetric objects. Our aim was to determine how well the symmetry estimators perform in typical industrial inspection tasks, where both the lighting and the background can be controlled, making the objects under view easy to separate on the basis of intensity from the background. Here, the intensity thresholds governing the separation were determined *manually, but with controlled lighting this should not be difficult to automate. Other forms of object/background separation, for example from range data, may also be used. Once the separation was accomplished, the center of mass was estimated by (18). The maximum radius R was then determined manually by inspecting the size of the object. This may also be automated by measuring the maximum distance of a pixel in the object from the center of mass, or it may be set as a constant if the size of all objects under consideration was known beforehand. The estimation procedure previously mentioned was then applied. The entire procedure took less than 2 minutes for each image, the time increasing over the previous set of experiments because not knowing R beforehand prevented us from using table lookups of the basis functions. Figure 2 shows a typical result with the estimated axis of symmetry superimposed.*

4. CONCLUSIONS

The aim of this paper is to develop optimal estimates of the parameters of linear phase signals from noisy measurements. When the noise is Gaussian, the estimators that are developed here are the maximum-likelihood estimators. They are applicable to both 1-D and 2-D data. In tests conducted on various synthetic and real images, accurate estimates were obtained of the inclinations of reflection axes.

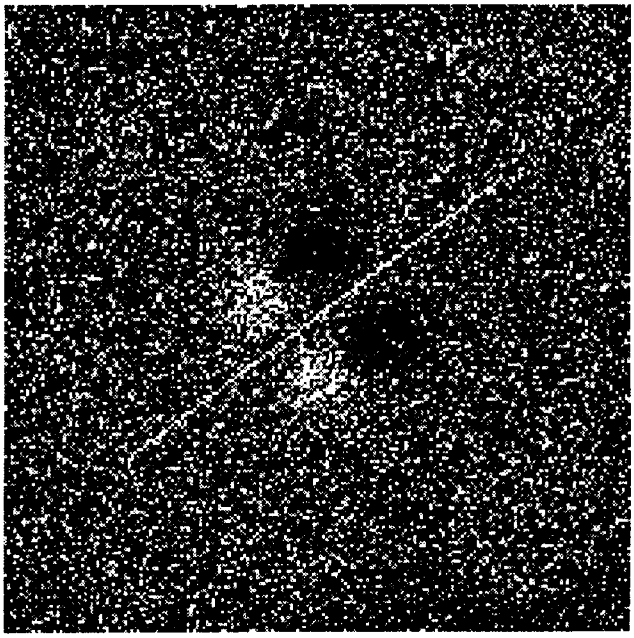

Figure 1: The synthetic image $g(x,y)$ with Gaussian noise added, with SNR=5 dB. The estimated axis of symmetry is shown superimposed.

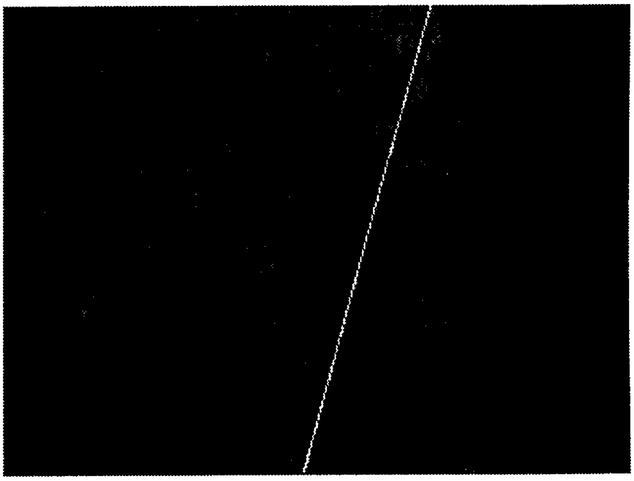

Figure 2: A pair of pliers shown with the estimated symmetry axis superimposed. The normalized error criterion is 0.11 with 9 coefficients.

5. REFERENCES

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