ACCURACY ANALYSIS OF ESTIMATION ALGORITHMS FOR PARAMETERS OF MULTIPLE POLYNOMIAL-PHASE SIGNALS

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ABSTRACT

The paper develops a method of error analysis for Fourier-transform based sinusoidal frequency estimation in the presence of nonrandom interferences. A general error formula is derived, and then specialized to the cases of additive and multiplicative interferences. Approximate error formulas are derived for the case of additive polynomial-phase interference. Finally, an application to error-analysis in estimating the parameters of multiple polynomial-phase signals is discussed in detail.

1. INTRODUCTION

The need to estimate the frequency of a (real or complex) sinusoidal waveform arises very often. When the signal is clean (free of interference of any kind) the problem is trivial. Interference introduces errors and renders the problem nontrivial. Interference, when exists, can be additive, multiplicative, or more complex.

In this paper we examine the problem of estimating the frequency of a sinusoid in the presence of *nonrandom* (deterministic) interference. The assumed continuous-time signal model is

$$y(t) = m(t)e^{j\omega_0 t} + a(t), \quad 0 \le t \le T, \qquad (1)$$

where $m(t)$ and $a(t)$ are unknown, but nonrandom functions, referred to as the multiplicative and additive interferences, respectively. The problem is to estimate the unknown frequency ω_0 and, in particular, to examine the accuracy of the estimate as a function of the pertinent properties of the functions $m(t)$ and $a(t)$.

When there is no specific knowledge about the nature of the interference, common sense and experience suggest using the Fourier transform (FT), and taking the frequency estimate as the point of maximum of the magnitude of the FT. For a clean signal, the point of maximum coincides with the true frequency ω_0. The behavior of the error in the point of maximum in the presence of interference is the subject matter of this paper.

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2. THE BASIC ERROR FORMULA

In this section we derive an error formula for the estimated frequency of a continuous-time complex sinusoid in the presence of additive and multiplicative interferences. The signal is given by (1) and its Fourier transform by $Y(\omega) = \int_0^T y(t)e^{-j\omega t}dt$. $Y(\omega)$ is assumed to have continuous derivatives up to a third order (in the sequel we will use primes to denote differentiations with respect to ω).

The estimated frequency $\hat{\omega}$ is defined as the point of global maximum of $|Y(\omega)|^2$ (or, as was explained in the Introduction, as some point of local maximum within a certain frequency range). If $a(t)$ and $m(t)$ are "sufficiently well-behaved" (in a sense made precise below), $\hat{\omega}$ will be "close" to ω_0. To find exactly how close, we will need the following lemma.

Lemma 1: Let $y = f(x)$ be twice continuously differentiable on the real line $\mathcal{R}$ and x_0 a point in $\mathcal{R}$. Assume that:

A: $f(x_0) \neq 0$ and $f'(x_0) \neq 0$;

B: the second derivative of $f(x)$ is bounded, that is, $|f''(x)| \le f''_{\max}$ for all $x \in \mathcal{R}$;

C: $[f'(x_0)]^2 \ge 2|f(x_0)|f''_{\max}$.

Denote

$$x_l = x_0 - \frac{2\frac{f(x_0)}{f'(x_0)}}{1 + \sqrt{1 - \frac{2f(x_0)f''_{\max}}{[f'(x_0)]^2}}}, \qquad (2)$$

$$x_u = x_0 - \frac{2\frac{f(x_0)}{f'(x_0)}}{1+\sqrt{1+\frac{2f(x_0)f''_{\max}}{[f'(x_0)]^2}}} \quad (3)$$

(the right hand sides exist by Assumptions A and C). Then there exists $\hat{x} \in [x_l, x_u]$ such that $f(\hat{x}) = 0$.
Proof: See [4].
Corollary: Let x_0 be a *fixed* point in $\mathcal{R}$ and $\{f_T(x), 0 < T < \infty\}$ a family of functions on $\mathcal{R}$ indexed by T, each satisfying Assumptions A, B, and C. Assume further that

$$\lim_{T\to\infty} \frac{f_T(x_0)f''_{T,\max}}{[f'_T(x_0)]^2} = 0. \quad (4)$$

Then the family of points $\hat{x}_T$ for which $f_T(\hat{x}_T) = 0$ is given by [1]

$$\hat{x}_T = x_0 - \frac{f_T(x_0)}{f'_T(x_0)}[1 + o(1)]. \quad (5)$$

The desired error formula for $\hat{\omega}$ is obtained by applying the lemma to the first derivative of the square-magnitude of $Y(\omega)$. Define $Z(\omega) = Y(\omega)[Y(\omega)]^*$. Assume that

$$\frac{Z'(\omega_0)Z'''(\omega)}{[Z''(\omega_0)]^2} = o(1). \quad (6)$$

Then

$$\hat{\omega} = \omega_0 - \frac{Z'(\omega_0)}{Z''(\omega_0)}[1 + o(1)]. \quad (7)$$

3. SPECIAL CASES OF THE ERROR FORMULA

We now examine in some more detail two special cases of the error formula, first the case of a purely additive interference, and then the case of a purely multiplicative interference.

3.1. Additive Interference

For a purely additive interference, $m(t) = 1$. Assume that $a(t)$ is bounded, say $|a(t)| \le B$ for all t. For every nonnegative integer k define $A_k(\omega) = \int_0^T t^k a(t)e^{-j\omega t}dt$. Obviously, $|A_k(\omega)|$ is uniformly bounded by $BT^{k+1}/(k+1)$. Assume further that $A_k(\omega_0) = o(T^{k+1})$. We get (see details in [4])

$$\begin{aligned}\hat{\omega} &= \omega_0 + \frac{12\mathrm{Im}\{A_1(\omega_0) - 0.5TA_0(\omega_0)\}}{T^3}[1 + o(1)] \\ &= \omega_0 + \frac{12\mathrm{Im}\left\{\int_0^T (t-0.5T)a(t)e^{-j\omega_0 t}dt\right\}}{T^3} \cdot \\ &\quad [1 + o(1)]. \end{aligned} \quad (8)$$

[1]The notations $o(\cdot)$ and $O(\cdot)$ are understood to be with respect to T as $T \to \infty$.

3.2. Multiplicative Interference

For a purely multiplicative interference, $a(t) = 0$. For every nonnegative integer k define

$$M_k(\omega) = \int_0^T t^k m(t)e^{-j(\omega-\omega_0)t}dt. \quad (9)$$

Then we have

$$\begin{aligned} Z'(\omega_0) &= 2\mathrm{Im}\{M_1(0)M_0^*(0)\}, \quad (10) \\ Z''(\omega_0) &= -2\mathrm{Re}\{M_2(0)M_0^*(0)\} + 2|M_1(0)|^2 \quad (11) \end{aligned}$$

To exemplify the case of multiplicative interference, consider a multiplicative interference of the form $m(t) = e^{jf_T(t)}$, where $\{f_T(t),\ 0 \le t \le T\}$ is $O(T^{-\epsilon})$ uniformly in t for some $\epsilon > 0$. Then we get, similarly to the case of additive interference,

$$\hat{\omega} = \omega_0 + \frac{12\int_0^T (t-0.5T)f_T(t)dt}{T^3}[1 + o(1)]. \quad (12)$$

4. ADDITIVE POLYNOMIAL-PHASE INTERFERENCE

In the remainder of this paper we will concern ourselves with additive interferences which are sums of polynomial-phase signals. A complex polynomial-phase signal is defined as

$$a(t) = \beta\exp\{j\phi(t)\}, \quad \phi(t) = \sum_{m=0}^{M} \alpha_m t^m, \quad (13)$$

where β is a real constant. To use the results of Sec. 3.1, we need to compute the integral

$$G = \beta\int_0^T (t-0.5T)e^{j[\phi(t)-\omega_0 t]}dt. \quad (14)$$

In the case of linear FM ($M = 2$), this integral can be expressed in a closed form, see [4]. When $M > 2$ there is no closed-form expression for this integral. However, a numerical approximation can be derived using the *principle of stationary phase*. By this principle, the main contribution to the integral is accumulated near the points where the argument of the corresponding trigonometric function is stationary, that is, its derivative is zero. Suppose the equation

$$\phi'(t) - \omega_0 = \sum_{m=1}^{M} m\alpha_m t^{m-1} - \omega_0 = 0 \quad (15)$$

has solutions $\{t_k, 0 \le k \le K-1\}$ in the interval $(0, T)$, where $K \ge 1$. Then

$$\begin{aligned} G \;\approx\; & \beta\sum_{k=0}^{K-1}\left[\frac{2\pi}{|\phi''(t_k)|}\right]^{1/2}(t_k - 0.5T) \\ & \exp\{j\,\mathrm{sign}[\phi''(t_k)][\phi(t_k) - \omega_0 t_k] + j0.25\pi\}, \end{aligned} \quad (16)$$

where

$$f''(t) = \sum_{m=2}^{M} m(m-1)\alpha_m t^{m-2}. \quad (17)$$

A *Mathematica* implementation of the approximation (16) is given in [4].

5. APPLICATION TO MULTIPLE POLYNOMIAL-PHASE SIGNALS

Suppose we are given a polynomial-phase signal, as in (13), and we wish to estimate the phase coefficients $\{\alpha_m\}$. This can be accomplished using the ***high-order ambiguity function*** (HAF), defined as

$$P_p[y;\omega,\tau] \triangleq \int_0^{T-(p-1)\tau} \mathcal{P}_p[y(t);\tau]e^{-j\omega t}dt, \quad (18)$$

where

$$\mathcal{P}_p[y(t);\tau] \triangleq \prod_{q=0}^{p-1}[y^{(*q)}(t+(p-1-q)\tau)]^{\binom{p-1}{q}}, \quad (19)$$

and

$$y^{(*q)}(t) \triangleq \begin{cases} y(t), & q \text{ even} \\ y^*(t), & q \text{ odd} \end{cases}. \quad (20)$$

$\mathcal{P}_p[y(t);\tau]$ will be called the pth-order HAF operator. Applying the Mth-order HAF to the signal (13) yields a spectral line at the frequency $\omega_0 = M!\tau^{M-1}\alpha_M$, thus enabling the estimation of α_M. After α_M is estimated, the degree of the phase polynomial is reduced by 1 through multiplication of the given signal by $e^{-j\alpha_M t^M}$ (this is called the ***phase-removal step***). Then the procedure is repeated with the $(M-1)$th-order HAF, etc.

Recently, Peleg and Friedlander [3], [2] have examined possible application of the HAF to sums of polynomial-phase signals, that is, to signals of the type

$$y(t) = \sum_{\ell=1}^{L} \beta_\ell \exp\{j\phi_\ell(t)\}, \quad \phi_\ell(t) = \sum_{m=0}^{M_\ell} \alpha_{\ell,m}t^m. \quad (21)$$

When the Mth-order HAF is applied to the signal (21), it yields a total of $L^{2^{M-1}}$ terms. If $M = M_\ell$ for some ℓ, there will be a spectral line at the frequency $M_\ell!\tau^{M-1}$ α_{ℓ,M_ℓ} (or more than one spectral line, if $M = M_\ell$ for more than one ℓ). The remaining terms will be polynomial-phase signals, so they can be regarded as additive polynomial-phase interferences. In principle, the L components can be treated one at a time. At the ℓth stage we apply the HAF algorithm M_ℓ times iteratively to the ℓth component, while regarding the other components as interferences (which amounts to ignoring them). Note that each phase-removal step reduces the order of the currently estimated component, but not of the interferences, so the interferences remain polynomial-phase signals throughout.

The tools developed in the previous section can be used for accuracy analysis of the HAF in estimating the parameters of sums of polynomial-phase signals. We will now describe the details of the analysis. We concentrate on the errors in estimating a single component, which we rename as $\beta\exp\{j\sum_{m=0}^{M}\alpha_m t^m\}$. We assume that $\beta = 1$ for convenience.

When the Mth order HAF (with $\tau = T/M$) is applied to the signal $y(t) = \exp\{j\sum_{m=0}^{M}\alpha_m t^m\} + a(t)$, the result is a signal of the form

$$P_M[y(t);T/M] = e^{j(\gamma_1 t+\gamma_0)} + \tilde{a}(t), \quad (22)$$

where $\tilde{a}(t)$ is a sum of polynomial-phase interferences, and γ_1, γ_0 are given in [4]. The interferences will cause the estimate $\hat{\gamma}_1$ to deviate from γ_1. Sections 3.1 and 4 show how to compute the asymptotic error resulting from a single polynomial-phase interference. The total asymptotic error is simply the sum of the individual errors. Thus, the analysis of the error in α_M is straightforward. Let us then continue to analyze the errors in the lower-order parameters.

Suppose we have already estimated the coefficients of orders $p+1$ through M, and denote the corresponding estimates by $\{\hat{\alpha}_m,\ p+1 \le m \le M\}$. Denote also

$$\delta_m \triangleq \hat{\alpha}_m - \alpha_m, \quad p+1 \le m \le M. \quad (23)$$

Let $z(t)$ be the signal obtained after removing from $y(t)$ the estimated phase terms of orders $p+1$ through M, that is,

$$\begin{aligned} z(t) &= \exp\left\{j\sum_{m=0}^{p}\alpha_m t^m - j\sum_{m=p+1}^{M}\delta_m t^m\right\} \\ &+ a(t)\exp\left\{-j\sum_{m=p+1}^{M}\hat{\alpha}_m t^m\right\}. \end{aligned} \quad (24)$$

Applying the pth-order HAF operator to $z(t)$ yields a signal of the form

$$P_p[z(t);T/p] = \exp\left\{j\sum_{i=0}^{M-p+1}\gamma_i t^i\right\} + \tilde{a}(t), \quad (25)$$

where $\tilde{a}(t)$ is a sum of polynomial-phase interferences. The paper [4] gives explicit formulas for the coefficients $\{\gamma_i\}$. As we see, the signal $P_p[z(t);T/p]$ has the form (up to a constant phase factor)

$$P_p[z(t);T/p] = \tilde{m}(t)e^{j\gamma_1 t} + e^{-j\gamma_0}\tilde{a}(t), \quad (26)$$

where

$$\tilde{m}(t) = \exp\left\{-j\sum_{i=2}^{M-p+1}\gamma_i t^i\right\}. \quad (27)$$

The error between $\hat{\gamma}_1$ and $p!(T/p)^{p-1}\alpha_p$ thus comes from three sources:

(i) The difference between γ_1 and $p!(T/p)^{p-1}\alpha_p$.

(ii) The additive interference $\tilde{a}(t)$.

(iii) The multiplicative interference $\tilde{m}(t)$.

The total error is asymptotically the sum of the three, so we can analyze them separately. Error (i) is given by $-\sum_{m=p+1}^{M} mT^{m-1}\eta_{m-1}\delta_m$. Error (ii) is treated as discussed before, that is, using the general additive-polynomial-phase-interference formula. It only remains to analyze error (iii). While doing so, we will also prove the following:

Claim: For all $1 \le m \le M$, the error δ_m is $O(T^{-m-1})$.

The proof is by induction, starting at $m = M$. The value of δ_M is only affected by error (ii). We have already seen that the frequency error due to additive polynomial-phase interference is $O(T^{-2})$. The coefficient α_M is proportional to the frequency by a factor $[(M!)(T/M)^{M-1}]^{-1}$, so δ_M is $O(T^{-M-1})$. Assume that the claim holds for $p+1 \le m \le M$. Using the induction hypothesis, we see that error (i) is $O(T^{-2})$. Error (ii) is $O(T^{-2})$ by the general result for additive polynomial-phase interference. It only remains to show that error (iii) is $O(T^{-2})$. It will then follow that $\hat{\gamma}_1 - \gamma_1$ is $O(T^{-2})$, hence δ_p is $O(T^{-p-1})$.

Observe from the induction hypothesis that γ_i is $O(T^{-i-1})$ for all $2 \le i \le M-p+1$. Therefore $\sum_{i=2}^{M-p+1} \gamma_i t^i$ is $O(T^{-1})$ uniformly in t for $0 \le t \le T$. The multiplicative interference (27) thus satisfies the condition of the second example in Sec. 3.2, with $\epsilon = 1$. We can therefore use the formula (12) to get

$$12(T/p)^{-3}\int_0^{T/p}(t-0.5T/p)f_T(t)dt = -\sum_{m=p+1}^{M} (T/p)^{m-1}\delta_m \sum_{i=2}^{m-p+1} \frac{6i}{(i+1)(i+2)}\binom{m}{i}\eta_{m-i}. \quad (28)$$

Since $\delta_m = O(T^{-m-1})$ by the induction hypothesis, the right side of (28) is $O(T^{-2})$, thus completing the proof.

We note that the expressions for errors (i) and (iii) can be combined to yield

$$\text{(i)} + \text{(iii)} = -\sum_{m=p+1}^{M} T^{m-1}\xi_{p,m}\delta_m, \quad (29)$$

where

$$\xi_{p,m} = p^{-m+1}\sum_{i=1}^{m-p+1}\frac{6i}{(i+1)(i+2)}\binom{m}{i} \sum_{k=0}^{p-1}\binom{p-1}{k}(-1)^{p-1-k}k^{m-i}. \quad (30)$$

Note also that $\xi_{p,p} = p^{-p+1}p!$ and δ_p is proportional to the error in $\hat{\gamma}_1$ by a factor $p!(T/p)^{p-1}$. Therefore we get, using (29) and collecting for $1 \le p \le M$,

$$\begin{bmatrix} \xi_{M,M} & 0 & \dots & 0 \\ \xi_{M-1,M} & \xi_{M-1,M-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{1,M} & \xi_{1,M-1} & \dots & \xi_{1,1} \end{bmatrix} \begin{bmatrix} T^{M-1} & 0 & \dots & 0 \\ 0 & T^{M-2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} \delta_M \\ \delta_{M-1} \\ \vdots \\ \delta_1 \end{bmatrix} = \begin{bmatrix} \epsilon_M \\ \epsilon_{M-1} \\ \vdots \\ \epsilon_1 \end{bmatrix}. \quad (31)$$

The quantities $\{\epsilon_p\}$ are the additive errors due to the polynomial-phase interferences. Recall that they are computed as follows. At the pth stage (except $p = M$) we let

$$a_p(t) = a(t)\exp\left\{-j\sum_{m=p+1}^{M}\alpha_m t^m\right\}. \quad (32)$$

Note that replacing $\hat{\alpha}_m$, as (24) implies, by α_m does not affect the asymptotic error. We then compute the corresponding $\tilde{a}_p(t)$ as the total additive interference in $z_p(t)$ (obtained from $y(t)$ by phase removal and pth-order HAF). We let $\gamma_0 = (p-1)!(T/p)^{p-1}\alpha_{p-1} + 0.5(p-1)p!(T/p)^p\alpha_p$, and finally compute ϵ_p using the additive polynomial-phase interference formula with $e^{-j\gamma_0}\tilde{a}_p(t)$. Once the vector of ϵ_p's have been computed we can solve (31) for the δ_m's.

6. REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1965.

[2] B. Friedlander and S. Peleg, "Multi Component Signal Analysis Using the Polynomial Transform," *26th Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, California, October 26–28, 1992.

[3] S. Peleg and B. Friedlander, "Multi Component Signal Estimation Using the Discrete Polynomial Phase Transform", *IEEE Trans. Aerospace Electronic Systems*, to appear.

[4] B. Porat and B. Friedlander, *Frequency Estimation in the Presence of Nonrandom Interference*, submitted for publication.