AMPLITUDE AND PHASE ESTIMATOR STUDY IN PRONY METHOD FOR NOISY EXPONENTIAL DATA

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ABSTRACT

The Prony method is a technique for modeling a data set of N samples using a linear combination of p exponentials ($N > p$) in a least square estimation procedure. In this paper, **a study of the amplitude and phase estimator** is presented. In particular, **theoretical** results are given for **bias and variance,** which are confirmed by simulations. **An optimal number of equations in the corresponding least square estimation procedure is derived**, minimizing estimation errors.

1. INTRODUCTION

Our problem is the accurate estimation of the parameters of exponentially damped sinusoidal signals from a data sequence of finite length. One of the estimation procedures is the Prony method [1] [2]. Prony's method is a technique for modeling sampled data as a linear combination of exponentials. The results obtained by using this method amount to an approximate fit of an exponential model by solving least square solution.

There are two basic steps in the Prony method. Step 1 first determines the linear prediction parameters that fit the available data. The roots of a polynomial formed from the linear prediction coefficients will then yield the estimates of damping factor and frequency of each of the exponential terms. Step 2 involves the solution of a second set of linear equations, also called the Van der Monde equations, to yield the estimates of exponential amplitude and initial phase.

Because of the presence of noise in the signal, the estimates of roots are biased, which means a bias in the damping and frequency terms [3] [4]. These estimation errors are then carried forward into the estimates of amplitude and phase. This is why, we analyze the behavior of the amplitude and phase estimator for a given set of approximated roots.

The paper is organized in the following way : In Sec. 2 the classical extended Prony method is recalled. The bias and variance of the amplitude and phase estimator which are derived from the approximations to the Van der Monde error matrix and the pseudo inverse of the Van der Monde matrix, are presented in Sec. 3. In Sec. 4 an optimal number of Van der Monde equations is deduced from the minimization of the mean square error (MSE) between the true amplitude and phase, and the estimates. Sec. 5 investigates the performance of this estimator, via numerical simulations. The conclusions are in Sec. 6.

2. PROBLEM STATEMENT

Consider the N samples of the observed data sequence, $y(n)$, which contains a sum of p exponentials, $x(n)$, in a background of independent, identically distributed (i.i.d.) complex white noise, $w(n)$ of variance σ_w^2 . So for $n = 0, \cdots, N-1$:

$$y(n) = x(n) + w(n) = \sum_{m=1}^{p} b_m Z_m^n + w(n) \qquad (1)$$

with $b_m = A_m e^{j\Theta_m}$ and $Z_m = e^{\alpha_m + j2\pi\tilde{f}_m}$, where A_m is the amplitude, Θ_m is the phase in radian, α_m is a damping factor, and $\tilde{f}_m$ is the normalized frequency. The frequency $\tilde{f}_m$ and damping factor α_m are found by rooting polynomial :

$$A(Z) = 1 + \sum_{k=1}^{p} a_k Z^{-k} = \prod_{m=1}^{p} (1 - Z_m Z^{-1}) \qquad (2)$$

where the a_k's are the coefficients of the recursive difference equation, for $n = p, \cdots, N-1$:

$$y(n) = -\sum_{k=1}^{p} a_k y(n-k) + w(n) + \sum_{k=1}^{p} a_k w(n-k) \quad (3)$$

Thus the extended Prony parameter estimation procedure reduces to an auto-regressive (AR) parameter

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estimation of an ARMA process by solving the least square solution of the modified Yule and Walker equations.

Once the $\widehat{Z}_k = e^{\widehat{\alpha}_k + j\widehat{\omega}_k}$ ($\widehat{}$ denoting the estimate) have been determined from the polynomial rooting, the exponential approximation $y(n)$ involving Eq(1) reduces to a set of linear equations in the unknown b_m parameters, expressible in matrix form as

$$\widehat{V}\widehat{\underline{b}} = \underline{y} = \underline{x} + \underline{w} \tag{4}$$

where

$$\widehat{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \widehat{Z}_1 & \widehat{Z}_2 & \cdots & \widehat{Z}_p \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{Z}_1^{M-1} & \widehat{Z}_2^{M-1} & \cdots & \widehat{Z}_p^{M-1} \end{bmatrix}$$

$$\widehat{\underline{b}} = \begin{bmatrix} \widehat{b}_1 & \widehat{b}_2 & \cdots & \widehat{b}_p \end{bmatrix}^T$$

$$\underline{\beta} = \begin{bmatrix} \beta(0) & \beta(1) & \cdots & \beta(N-1) \end{bmatrix}^T$$

$$\beta = x, w \text{ or } y$$

The solving of the $M \times p$ Van der Monde system (4) leads to the amplitude and phase estimation :

$$\widehat{\underline{b}} = \left(\widehat{V}^H\widehat{V}\right)^{-1}\widehat{V}^H\underline{y} = \left(\widehat{V}^H\widehat{V}\right)^{-1}\widehat{V}^H(\underline{x}+\underline{w}) \tag{5}$$

Eq(5) shows that noise perturbation is introduced in the $\underline{y}$ term, and also in the Van der Monde pseudo inverse $\left(\widehat{V}^H\widehat{V}\right)^{-1}\widehat{V}^H$ due to the estimation errors of the roots $\widehat{Z}_k$. Consequently, the study of this estimator causes quite a few problems. So let us assume that the estimated roots are given once and for all, independently of each signal realization.

3. BIAS AND VARIANCE EXPRESSIONS

As the term $\widehat{V}^\sharp = \left(\widehat{V}^H\widehat{V}\right)^{-1}\widehat{V}^H$ is considered constant, then, from Eq(5), bias and variance expressions are :

- for bias,

$$E(\widehat{\underline{b}}) - \underline{b} = \left(\widehat{V}^H\widehat{V}\right)^{-1}\widehat{V}^H E(\underline{y}) - \underline{b}$$

$$E(\widehat{\underline{b}}) - \underline{b} = \left(\widehat{V}^H\widehat{V}\right)^{-1}\widehat{V}^H \underline{x} - \underline{b} = \widehat{V}^\sharp\underline{x} - \underline{b} \tag{6}$$

- for variance,

$$Var(\widehat{\underline{b}}) = E\left(\left|\widehat{\underline{b}} - E(\widehat{\underline{b}})\right|^2\right)$$
$$= \widehat{V}^\sharp E\left(\underline{w}.\underline{w}^H\right)\left(\widehat{V}^\sharp\right)^H$$

from the assumption $E\left(\underline{w}.\underline{w}^H\right) = \sigma_w^2.I$, I being the identity matrix, we obtain :

$$Var(\widehat{\underline{b}}) = \sigma_w^2\left(\widehat{V}^H\widehat{V}\right)^{-1} \tag{7}$$

Taking into account that $\underline{x}$ is an exponential vector which can be written as $\underline{x} = V\underline{b}$, and denoting the Van der Monde error matrix by $\Delta V = \widehat{V} - V$, Eq(6) gives

$$E(\widehat{\underline{b}}) = \widehat{V}^\sharp V\underline{b} = \underline{b} - \widehat{V}^\sharp \Delta V \underline{b} \tag{8}$$

Let us now consider the following approximations to ΔV and $\widehat{V}^\sharp$.

3.1. Approximation to the Van der Monde error matrix ΔV

Due to the fact that matrix $\widehat{V}$ consists of terms $\widehat{Z}_m$ to the nth power which can be approximated by a first order series expansion, provided that $\frac{|\Delta Z_m|}{|Z_m|} \ll 1$:

$$\left(\widehat{Z}_m\right)^n = (Z_m + \Delta Z_m)^n \approx Z_m^n + n.Z_m^{n-1}\Delta Z_m \tag{9}$$

where $\Delta Z_m = \widehat{Z}_m - Z_m$ is the root error, it follows that :

$$\Delta V = \widehat{V} - V \approx$$
$$\begin{bmatrix} 0 & \cdots & 0 \\ \Delta Z_1 & \cdots & \Delta Z_p \\ 2\Delta Z_1 Z_1 & \cdots & 2\Delta Z_p Z_p \\ \vdots & \ddots & \vdots \\ (M-1)\Delta Z_1 Z_1^{M-2} & \cdots & (M-1)\Delta Z_p Z_p^{M-2} \end{bmatrix} \tag{10}$$

3.2. Approximation to the pseudo inverse $\widehat{V}^\sharp$

The matrix $\widehat{V}^H\widehat{V}$ has the following structure :

$$\widehat{V}^H\widehat{V} = \begin{bmatrix} \widehat{\underline{V}}_1^H\widehat{\underline{V}}_1 & \widehat{\underline{V}}_1^H\widehat{\underline{V}}_2 & \cdots & \widehat{\underline{V}}_1^H\widehat{\underline{V}}_p \\ \widehat{\underline{V}}_2^H\widehat{\underline{V}}_1 & \widehat{\underline{V}}_2^H\widehat{\underline{V}}_2 & \cdots & \widehat{\underline{V}}_2^H\widehat{\underline{V}}_p \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\underline{V}}_p^H\widehat{\underline{V}}_1 & \widehat{\underline{V}}_p^H\widehat{\underline{V}}_2 & \cdots & \widehat{\underline{V}}_p^H\widehat{\underline{V}}_p \end{bmatrix} \tag{11}$$

where $\widehat{\underline{V}}_m = \begin{bmatrix} 1 & \widehat{Z}_m & \widehat{Z}_m^2 & \cdots & \widehat{Z}_m^{M-1} \end{bmatrix}^T$, $1 \leq m \leq p$. Comparison of the diagonal with the off diagonal terms, enables us to say that diagonal terms are preponderant. Their expressions are :

$$\widehat{\underline{V}}_m^H\widehat{\underline{V}}_m = \begin{cases} \frac{1-e^{2M\widehat{\alpha}_m}}{1-e^{2\widehat{\alpha}_m}} & \text{if } \widehat{\alpha}_m \neq 0 \\ M & \text{if } \widehat{\alpha}_m = 0 \end{cases}$$
$$\widehat{\underline{V}}_m^H\widehat{\underline{V}}_k = \frac{1-e^{M(\widehat{\alpha}_m+\widehat{\alpha}_k)}e^{jM(\widehat{\omega}_m-\widehat{\omega}_k)}}{1-e^{(\widehat{\alpha}_m+\widehat{\alpha}_k)}e^{j(\widehat{\omega}_m-\widehat{\omega}_k)}} \tag{12}$$

If we suppose that damping factors approximate to zero, this amounts to comparing $\left|\underline{\widehat{V}}_m^H \underline{\widehat{V}}_k\right|$ to M. $\left|\underline{\widehat{V}}_m^H \underline{\widehat{V}}_k\right|$ then becomes equal to M when $\widehat{\omega}_m = \widehat{\omega}_k$ or else $\widehat{\omega}_m - \widehat{\omega}_k = \pi$ (for $\widehat{\alpha}_m$ and $\widehat{\alpha}_k \approx 0$), but these cases are fanciful. In the end, off-diagonal terms are insignificant compared to diagonal terms. So to approximate the inverse of $\widehat{V}^H\widehat{V}$ amounts to inverting a diagonal matrix :

$$\left(\widehat{V}^H\widehat{V}\right)^{-1} \approx$$

$$\begin{bmatrix} \frac{1-e^{2\widehat{\alpha}_1}}{1-e^{2M\widehat{\alpha}_1}} & 0 & \cdots & 0 \\ 0 & \frac{1-e^{2\widehat{\alpha}_2}}{1-e^{2M\widehat{\alpha}_2}} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1-e^{2\widehat{\alpha}_p}}{1-e^{2M\widehat{\alpha}_p}} \end{bmatrix} \quad (13)$$

For nearly zero damping factors, matrix (13) can be simplified to :

$$\left(\widehat{V}^H\widehat{V}\right)^{-1} \approx \frac{1}{M} I$$

which implies that the approximation to the Van der Monde pseudo inverse is :

$$\widehat{V}^\sharp \approx \frac{1}{M}\widehat{V}^H \quad (14)$$

In what follows, we investigate the particular case of the undamped exponential signal. Thus estimated damping factors are close to zero.

3.3. Approximations to $E(\underline{\widehat{b}})$ and $Var(\underline{\widehat{b}})$

It follows from Eq(8), Eq(7), and approximations (10) and (14) that,

- for bias,

$$E(\widehat{b}_k) - b_k \approx -\tfrac{M-1}{2}\Delta Z_k \widehat{Z}_k^* b_k - \tfrac{1}{M}\sum_{i=1;i\neq k}^{p} \Delta Z_i \widehat{Z}_k^* G(Z_i \widehat{Z}_k^*) b_i \quad (15)$$

$$\text{with } G(z) = \frac{1 + (M-1)z^M - Mz^{M-1}}{(1-z)^2} \quad (16)$$

- for variance,

$$Var(\widehat{b}_k) \approx \sigma_w^2 \frac{1 - e^{2\widehat{\alpha}_k}}{1 - e^{2M\widehat{\alpha}_k}} \approx \frac{\sigma_w^2}{M} \quad (17)$$

According to these results, both bias and variance depend on M. Bias can be shown to consist of two terms :

- the first, $-\frac{M-1}{2}\Delta Z_k \widehat{Z}_k^* b_k$, which is the main term, is proportional to M.
- the second, $-\frac{1}{M}\sum_{i=1;i\neq k}^{p} \Delta Z_i \widehat{Z}_k^* G(Z_i \widehat{Z}_k^*) b_i$, produces oscillations around the first term.

From Eq(17) we can see that when M increases variance decreases. The asymptotic variance for the term $\widehat{b}_k$ is

$$Var(\widehat{b}_k) \underset{M\to+\infty}{\longrightarrow} \sigma_w^2 \left(1 - e^{2\widehat{\alpha}_k}\right) \quad (18)$$

4. OPTIMAL NUMBER OF VAN DER MONDE EQUATIONS

An optimal number $M_{k_{opt}}$ that minimizes MSE d_k can then be determined :

$$\begin{aligned} d_k &= E\left(\left|\widehat{b}_k - b_k\right|^2\right) \\ &= Var(\widehat{b}_k) + \left|E(\widehat{b}_k) - b_k\right|^2 \end{aligned} \quad (19)$$

Substituting $Var(\widehat{b}_k)$ from Eq(17) into Eq(19) and substituting $E(\widehat{b}_k) - b_k$ from the first term in Eq(15) into Eq(19), gives :

$$d_k \approx \frac{\sigma_w^2}{M} + \frac{(M-1)^2}{4}\left|\Delta Z_k \widehat{Z}_k^*\right|^2 |b_k|^2$$

The optimal number $M_{k_{opt}}$ is found by deriving d_k with respect to M and equating the derivative to zero :

$$\frac{\partial d_k}{\partial M} \approx -\frac{\sigma_w^2}{M^2} + \frac{M-1}{2}\left|\Delta Z_k \widehat{Z}_k^*\right|^2 |b_k|^2$$

$$M_{k_{opt}} \simeq \sqrt[3]{\frac{2\sigma_w^2}{|b_k|^2 \left|\Delta Z_k \widehat{Z}_k^*\right|^2}} \quad (20)$$

Note that in Eq(20) if $\Delta Z_k = 0$ (when poles are exactly known), then $M_{k_{opt}} \to +\infty$ in order to reduce the noise effects on the solution $\widehat{b}_k$. Of course, each $\widehat{b}_k$ leads to a different $M_{k_{opt}}$. Hence, a number M_{opt} has to be chosen, such that it minimizes MSE $d = \frac{1}{p}\sum_{k=1}^{p} d_k$, deriving d with respect to M and equating the derivative to zero yield :

$$M_{opt} \simeq \sqrt[3]{\frac{2\sigma_w^2}{\frac{1}{p}\sum_{k=1}^{p} |b_k|^2 \left|\Delta Z_k \widehat{Z}_k^*\right|^2}} \quad (21)$$

5. SIMULATION RESULTS

To illustrate these theoretical results, let us consider an example of a signal $x(n) = \cos(0.6n) + 0.5\cos(n+1)$ and $\sigma_w^2 = 0.4$ ($SNR \simeq 4.5dB$) for a modeling order $p = 10$. The results are summarized in the following table :

True parameters	$\omega_1 = -\omega_2 = 0.6$ $\alpha_1 = \alpha_2 = 0$ $\omega_3 = -\omega_4 = 1$ $\alpha_3 = \alpha_4 = 0$	$b_1 = 0.5$ $b_2 = 0.5$ $b_3 = 0.25e^{j}$ $b_4 = 0.25e^{-j}$
Estimated parameters	$\widehat{\omega}_1 = -\widehat{\omega}_2 = 0.5993$ $\widehat{\alpha}_1 = \widehat{\alpha}_2 = -9.4643.10^{-4}$ $\widehat{\omega}_3 = -\widehat{\omega}_4 = 1.0035$ $\widehat{\alpha}_3 = \widehat{\alpha}_4 = -0.0073$	
Optimal number M	$M1_{opt} = M2_{opt} = 132$ $M3_{opt} = M4_{opt} = 58$ $M_{opt} = 72$	

Fig. 1 and 2 present the modulus and phase behavior of $E(\widehat{b}_1)$ as a function of M, using Eq(6) and the approximation given by Eq(15).

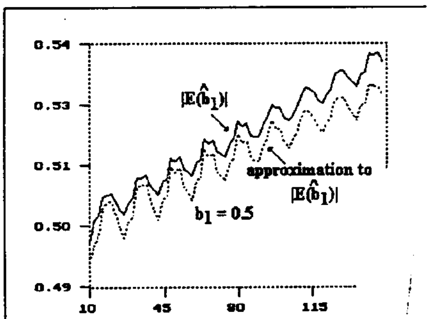

Fig. 1. Modulus of $E(\widehat{b}_1)$ ∖

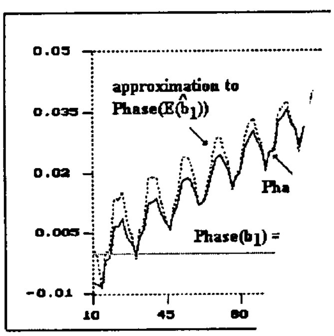

Fig. 2. Phase of $E(\widehat{b}_1)$

These figures illustrate the approximation quality of Eq(15) and hilight the **increasing behavior of $E(\widehat{b}_1)$ as a function of the Van der Monde equation number M.**

Fig. 3 presents variances of $\widehat{b}_1$ and $\widehat{b}_3$, as a function of M, according to Eq(7).

In Fig. 4 MSE d_1,d_2 and d are ploted versus the equation number M. The minima observed in this figure correspond to the values derived from Eq(20) and Eq(21).

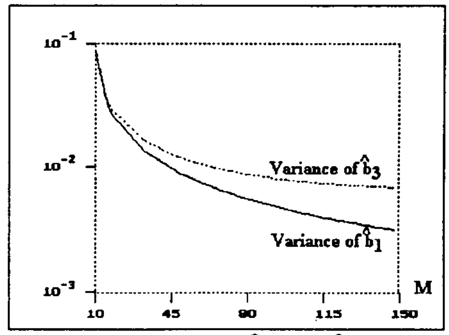

Fig. 3. Variances of $\widehat{b}_1$ and $\widehat{b}_3$ v.s. M

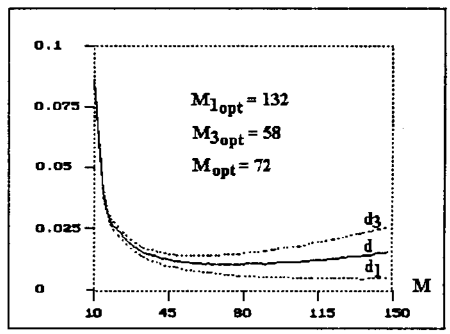

Fig. 4. Quadratic errors $d1$, $d2$ and d v.s. M

6. CONCLUSION

It is widely admitted that the reduction of the noise effect on the Van der Monde solution leads to a choice of M as large as possible. **One of the main results of our analysis is that, surprisingly, there exists an optimal number of equations M.**

7. REFERENCES

[1] Marple S.L.Jr. : Digital Spectral Analysis with Applications. Prentice-Hall. Inc., chap. 11.

[2] Kay S.M and Marple S.L.Jr : Spectrum Analysis-A Modern Perspective. Proc. IEEE, vol. 69, no. 11, pp 1380-1419, November 1981.

[3] Kay S.M : The Effects of Noise on the Autoregressive Spectral Estimator. IEEE ASSP, vol 27, no 5, pp 478-485, October 1979.

[4] Kumaresan R. and Tufts D.W. : Estimating the Parameters of Exponentially Damped Sinusoids and Pole-Zero Modeling in Noise. IEEE ASSP, vol 30, no 6, pp 833-840, December 1982.