ESTIMATION AND STATISTICAL ANALYSIS FOR EXPONENTIAL POLYNOMIAL SIGNALS

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ABSTRACT

In this paper we approximate arbitrary complex signals by modeling both the logarithm of the amplitude and the phase of the complex signal as finite-order polynomials in time. We refer to a signal of this type as an Exponential Polynomial Signal (EPS). We propose an algorithm to estimate any desired coefficient for this signal model. We also show how the mean-squared error of the estimate can be determined by using a first-order perturbation analysis. A Monte Carlo simulation is used to verify the validity of the perturbation analysis. The performance of the algorithm is illustrated by comparing the mean-squared error of the estimate to the Cramer-Rao bound for a particular example.

1. INTRODUCTION

It is well-known that an arbitrary complex signal can be represented by its magnitude and phase. In this paper we model complex signals by approximating the phase of the signal as a finite-order Taylor expansion in time. Further, we model the logarithm of the time-varying amplitude of the signal as a finite-order Taylor expansion as well. The types of signals that we consider in this paper arise in various applications such as geophysical phenomena [1] and speech processing [2].

In this paper, we consider observing a complex signal, s_n, in circular additive white Gaussian noise, w_n. That is, suppose we observe

$$y_n = s_n + w_n, \tag{1}$$

where n ranges from $1, 2, ..., N$. The signal, s_n, is modeled as being an EPS. Specifically,

$$s_n = \exp\left(a_0 + a_1 n + a_2 \frac{n^2}{2!} + ... + a_M \frac{n^M}{M!}\right), \tag{2}$$

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where the coefficients of this Taylor expansion are unknown complex parameters. The real parts of the coefficients specify the envelope of the signal, while the imaginary parts of the coefficients specify the phase of the signal.

In a recent paper [3], the authors presented a new method to estimate the highest-order coefficient, a_M, from the noisy observation, y_n. In [3], it was also shown that this new algorithm is related to an algorithm presented by Peleg [4] to estimate constant-amplitude polynomial phase signals. In this paper, we extend the concepts of [3] to determine an estimate for an arbitrary coefficient a_k by the means of analytically solving an optimization problem that depends only upon the observation and the desired coefficient. We also give a method to determine the mean-squared error of our estimate for a_k by using a first-order perturbation analysis. Some of the results in this paper are stated and not proved. The reader is directed to [5] for the detailed proofs.

2. ESTIMATION ALGORITHM

In order to determine a closed form expression for a_k, we consider taking the kth finite-difference of the logarithm of both sides of Eq. (2). The resulting equality is expressed as

$$\nabla_{\tau_k} \nabla_{\tau_{k-1}} \cdots \nabla_{\tau_1} \ln s_n = \sum_{i=k}^{M} a_i z_n^{k/i}, \tag{3}$$

where the finite-difference is defined by $\nabla_\tau f_n = f_n - f_{n-\tau}$. The function $z_n^{k/i}$ is defined as

$$z_n^{k/i} = \nabla_{\tau_k} \cdots \nabla_{\tau_1} \frac{n^i}{i!}.$$

The kth finite-difference of the logarithm of s_n can be determined recursively. Specifically,

$$\nabla_{\tau_k}\nabla_{\tau_{k-1}}\cdots\nabla_{\tau_1}\ln s_n = \ln\frac{g_n^k(\gamma_k)}{h_n^k(\gamma_k)} \tag{4}$$

where the recursions are

$$g_n^m(\gamma_m) = g_n^{m-1}(\gamma_{m-1})h_{n-\tau_m}^{m-1}(\gamma_{m-1}) \tag{5}$$

$$h_n^m(\gamma_m) = h_n^{m-1}(\gamma_{m-1})g_{n-\tau_m}^{m-1}(\gamma_{m-1}). \tag{6}$$

These recursions are initialized by choosing $g_n^0(1) = s_n$, and $h_n^0(1) = 1$. The recursive equations are applied k times, by setting m to $m = 1, 2, \cdots, k$. The function γ_k represents the product of the delay parameters. That is, $\gamma_k = \tau_1\tau_2\cdots\tau_k$. Note that the functions $g_n^m(\gamma_m)$ and $h_n^m(\gamma_m)$ depend upon all of the delay parameters individually rather than the product of the delay parameters as suggested by the argument of the function.

From Eq. (3), we note that the finite-differencing has eliminated the dependence on the parameters from a_0 to a_{k-1}. That is, this equation depends upon only $M - k + 1$ complex signal parameters. To determine a system of equations that is a function of these parameters we take the $k, k+1, \cdots, M$ finite-difference of the logarithm of the signal. By equating Eqs. (3) and (4), we state these equations as

$$g_n^m(\gamma_m) - \prod_{i=m}^{M}\exp\left(a_i z^{m/i}\right)h_n^m(\gamma_m) = 0 \tag{7}$$

for $m = k, k+1, \cdots, M$.

By solving this system of equations, we determine that the coefficient a_k satisfies

$$g^{k/M}(\gamma_k) - \exp(a_k\gamma_k)h^{k/M}(\gamma_k) = 0. \tag{8}$$

The superscripts k/M is used to denote that we are estimating a_k when the order of the polynomial of the signal is M. The functions in Eq. (8) are determined recursively by

$$g_n^{m/M}(\gamma_m) = g_n^m(\gamma_m)\bigotimes_{i=m+1}^{M} h^{i/M}(z_n^{m/i})$$

$$h_n^{m/M}(\gamma_m) = h_n^m(\gamma_m)\bigotimes_{i=m+1}^{M} g^{i/M}(z_n^{m/i}).$$

These equations are evaluated successively by letting $m = M-1, M-2, \cdots, k$. The recursion is initialized by noting that $g^{M/M}(\gamma_M) = g^M(\gamma_M)$ and $h^{M/M}(\gamma_M) = h^M(\gamma_M)$. The symbol $\otimes$ is used to denote the Kronecker product. In general, the functions $g_n^{m/M}(\gamma_m)$ and $h_n^{m/M}(\gamma_m)$ are column vectors for each n. Larger dimensional column vectors, $g^{m/M}(\gamma_m)$ and $h^{m/M}(\gamma_m)$, are obtained by stacking each of the smaller dimensional column vectors.

We obtain the least squares estimate of a_k by replacing the noise-free signal with the observation in Eq. (8) and minimizing the sum of the squares of the errors. That is,

$$\min_{a_k}\left\|\bar{g}^{k/M}(\gamma_k) - \exp(a_k\gamma_k)\bar{h}^{k/M}(\gamma_k)\right\|^2. \tag{9}$$

where we have used a bar over the vector to represent that the vector depends upon the observation rather than the noise-free signal. By analytically solving Eq. (9), we determine that the least squares estimate for a_k is expressed as

$$\hat{a}_k = \frac{1}{\gamma_k}\ln\frac{r^{k/M}(\gamma_k)}{v^{k/M}(\gamma_k)} \tag{10}$$

where

$$r^{k/M}(\gamma_k) = \left(\bar{h}^{k/M}(\gamma_k)\right)^H \bar{g}^{k/M}(\gamma_k) \tag{11}$$

$$v^{k/M}(\gamma_k) = \left\|\bar{h}^{k/M}(\gamma_k)\right\|^2 \tag{12}$$

By using some well-known properties of the Kronecker product [6], we express a recursion for computing the quantities in Eqs. (11) and (12). That is,

$$r^{k/M}(\gamma_k) = \sum_n \bar{h}_n^k(\gamma_k)^*\bar{g}_n^k(\gamma_k)\prod_{m=k+1}^{M} r^{m/M}(z_n^{k/m})^* \tag{13}$$

$$u^{k/M}(\gamma_k) = \sum_n \left|\bar{g}_n^k(\gamma_k)\right|^2\prod_{m=k+1}^{M} v^{m/M}(z_n^{k/m}) \tag{14}$$

$$v^{k/M}(\gamma_k) = \sum_n \left|\bar{h}_n^k(\gamma_k)\right|^2\prod_{m=k+1}^{M} u^{m/M}(z_n^{k/m}) \tag{15}$$

Note that the scalar functions of Eqs. (13-15) do not have to be recalculated for estimating each signal parameter.

An alternative estimation algorithm can be obtained by first dividing Eq. (8) by $\exp(a_k\gamma_k)$ before replacing the signal with the observation and minimizing the norm of the error. We refer to the estimate obtained in this fashion as the backwards least squares estimate. This estimate is given by

$$\hat{a}_k = \frac{1}{\gamma_k}\ln\frac{u^{k/M}(\gamma_k)}{r^{k/M}(\gamma_k)^*}.$$

3. STATISTICAL ANALYSIS

In this section, we derive an approximation for the variance for the estimate of the coefficient $\hat{a}_k$ by using a first-order perturbation analysis. That is, we expand the estimate $\hat{a}_k$ in a first-order Taylor expansion about the noise-free signal and then use this expansion to compute the variance. A similar method could be used to derive the bias of the estimate by making use of a second-order Taylor expansion.

Since the estimate $\hat{a}_k$ is a function of both the observation and the conjugate of the observation, we expand about both the noise-free signal and its conjugate. However, it can be shown that the derivative of $\hat{a}_k$ with respect to the conjugate of the observation evaluated when the observation is equal to the true signal is identically zero. Therefore, we can express the first-order Taylor expansion of the estimate as

$$\hat{a}_k = a_k + \sum_{i=1}^{N} \left. \frac{\partial \hat{a}_k}{\partial y_i} \right|_{y=s} w_i. \quad (16)$$

The variance of the complex estimate would then be given by

$$var(\hat{a}_k) = \sigma^2 \sum_{i=1}^{N} \left| \frac{\partial \hat{a}_k}{\partial y_i} \right|_{y=s}^2. \quad (17)$$

The variances for either the real or imaginary part of the estimate would be equal to each other and equal to one half of the variance of the complex estimate that is given in Eq. (17).

The derivative is determined by differentiating Eq. (10), that is

$$\frac{\partial \hat{a}_k}{\partial y_i} = \frac{1}{\gamma_k} \left(\frac{\frac{\partial r^{k/M}(\gamma_k)}{\partial y_i}}{r^{k/M}(\gamma_k)} - \frac{\frac{\partial v^{k/M}(\gamma_k)}{\partial y_i}}{v^{k/M}(\gamma_k)} \right). \quad (18)$$

The derivatives stated in Eq. (18) can be computed recursively using four equations. Specifically, we differentiate Eqs. (13-15) and the conjugate of Eq. (13) with respect to the observation. For example, the derivative of Eq. (13) is

$$\frac{\partial r^{k/M}(\gamma_k)}{\partial y_i} = \sum_n \bar{h}_n^k(\gamma_k)^* \frac{\partial \bar{g}_n^k(\gamma_k)}{\partial y_i} \prod_{m=k+1}^{M} r^{m/M}(z_n^{k/m})^*$$
$$+ \sum_n \bar{h}_n^k(\gamma_k)^* \bar{g}_n^k(\gamma_k) \sum_{l=k+1}^{M} \frac{\partial r^{l/M}(z_n^{k/l})^*}{\partial y_i}$$
$$\prod_{\substack{m=k+1 \\ m \neq l}}^{M} r^{m/M}(z_n^{k/m})^*.$$

We will also be needing the derivatives of Eqs. (5) and (6). Although these derivatives can be obtained by differentiating their recursive definitions, we find it beneficial to differentiate closed-form expressions of these functions. Specifically, we express these functions as products of delayed versions of the signal. That is,

$$g_n^k(\gamma_k) = \prod_{m=1}^{L} s_{n-\alpha_m}$$
$$h_n^k(\gamma_k) = \prod_{m=1}^{L} s_{n-\beta_m}$$

where $L = 2^{k-1}$. The vectors $\alpha^k = [\alpha_1 \ \cdots \ \alpha_L]$ and $\beta^k = [\beta_1 \ \cdots \ \beta_L]$ are determined recursively from the delay parameters. That is,

$$\alpha^m = [\alpha^{m-1} \quad \beta^{m-1} + \tau_m]$$

$$\beta^m = [\beta^{m-1} \quad \alpha^{m-1} + \tau_m],$$

where the recursion is initialized by choosing $\alpha^1 = 0$ and $\beta^1 = \tau_1$. The derivatives of $g^k(\gamma_k)$ and $h^k(\gamma_k)$ are stated as

$$\frac{\partial g^k(\gamma_k)}{\partial s_i} = \frac{1}{s_i} A^i g^k(\gamma_k) \quad (19)$$

$$\frac{\partial h^k(\gamma_k)}{\partial s_i} = \frac{1}{s_i} B^i h^k(\gamma_k). \quad (20)$$

where A^i and B^i are diagonal matrices. Specifically, the diagonal elements of these matrices are

$$A_n^i = \sum_{k=1}^{L} \delta_{n-\alpha_k-i}$$
$$B_n^i = \sum_{k=1}^{L} \delta_{n-\beta_k-i}.$$

Thus, we have presented a method to determine the variance for an arbitrary coefficient $\hat{a}_k$. For example, we use this procedure to determine that the variance of the highest-order coefficient, $\hat{a}_M$, is

$$var(\hat{a}_M) = \sigma^2 \sum_{i=1}^{N} \left(h^H \frac{(A^i - B^i)}{\gamma_M |s_i| \, \|h\|^2} h \right)^2. \quad (21)$$

Similarly, we can derive the bias for $\hat{a}_M$ as

$$bias(\hat{a}_M) = \sigma^2 \sum_{i=1}^{N} \frac{h^H (A^i - B^i)}{\gamma_M |s_i|^2 \, \|h\|^2} \left(I - \frac{hh^H}{\|h\|^2} \right) B^i h \quad (22)$$

In Eqs. (21) and (22), we represent $h^M(\gamma_M)$ by h.

An important observation from Eqs. (21) and (22) is that the bias and the variance of $\hat{a}_M$ depends only upon the amplitude of the signal (the real parts of the signal parameters) and not the phase of the signal (the imaginary parts of the signal parameters). Also note that from Eq. (22), the bias of $\hat{a}_M$ is purely real. That is, by using a second-order Taylor expansion of the estimate, we deduce that the highest-order phase coefficient is unbiased. Since the mean-squared error (MSE) is defined as the variance plus the absolute bias squared of the estimate, we note that any differences between the MSE of the $Re(a_M)$ and the $Im(a_M)$ is due to the bias of the $Re(a_M)$.

4. NUMERICAL SIMULATION

Here, we compare the MSE of the estimates obtained from a Monte Carlo simulation and the perturbation analysis to the Cramer-Rao lower bound (CRB) for a particular example. A derivation for the CRB for this model is shown in [3]. For our example, we considered a chirp with a Gaussian envelope where the signal parameters are given by $a_0 = -6 + 2j$, $a_1 = (4.8 + 2j) * 10^{-2}$, and $a_2 = (-2 + 4j) * 10^{-4}$. The number of samples was chosen to be $N = 500$. In Fig. 1, we show the MSE for the estimate of a_2 versus the second delay parameter τ_2 at a 20 db peak signal-to-noise ratio. The MSE using the perturbation analysis is compared to a Monte Carlo simulation with 1000 runs and to the CRB. The first delay parameter was taken to be $\tau_1 = 77$. From this figure, we chose $\tau_2 = 93$ to minimize the MSE. We observe from this figure that at low signal-to-noise ratios the choice of the delay parameters can have a significant effect on the bias of the amplitude parameters.

Using $\tau_1 = 77$ and $\tau_2 = 93$, we estimated a_2 in a Monte Carlo simulation with 300 runs and compared the MSE to the CRB in Fig. 2 using various signal-to-noise ratios. Also shown in Fig. 2 is a comparison of the MSE for a_1 to the CRB. For the estimation of a_1, we used the backwards least squares estimate and a delay parameter of $\tau_1 = 60$. Numerical simulations suggest that the bias of the estimate is reduced by using the backwards least squares estimate when the real part of the unknown coefficient is positive.

The results shown in these figures illustrate that the MSE for the estimates came close to achieving the CRB at high signal-to-noise ratios for a particular example. Also note that the Monte Carlo simulations confirm the validity of the perturbation analysis.

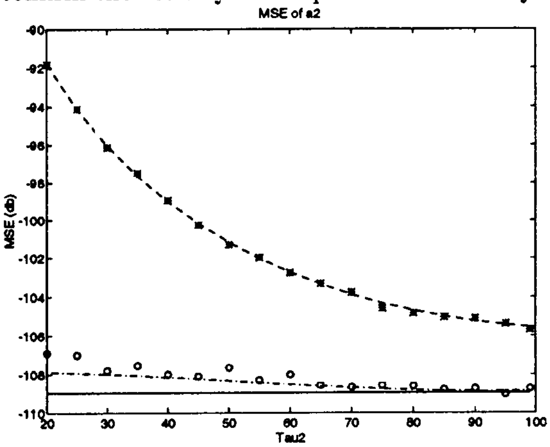

Fig. 1. Monte Carlo simulation: $Re(a_2) \Leftrightarrow (*)$ and $Im(a_2) \Leftrightarrow (0)$. Perturbation analysis: $Re(a_2) \Leftrightarrow (--)$ and $Im(a_2) \Leftrightarrow (-\cdot)$. CRB: $Re(a_2)$ and $Im(a_2)$ represented by solid line.

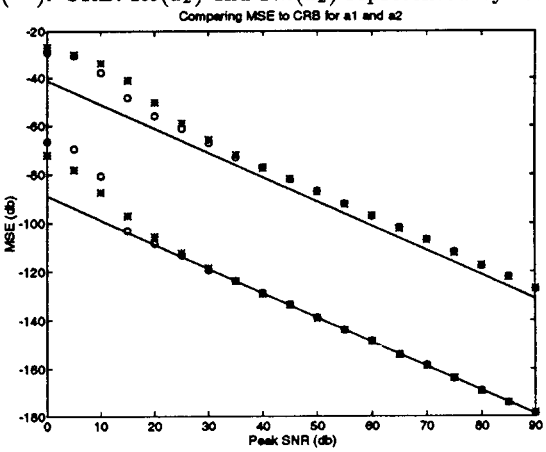

Fig. 2. The MSE for a_1 and a_2 are compared to the CRB (solid lines). The top curves represent a_1, while the bottom curves represent a_2. Monte Carlo simulation: $Re(a_2) \Leftrightarrow (*)$ and $Im(a_2) \Leftrightarrow (0)$.

5. REFERENCES

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