CONSTRAINED LEAST-SQUARES ESTIMATION OF A RANDOM AMPLITUDE SINUSOID

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ABSTRACT

In this paper, a new method for estimating the frequency of a random amplitude sinusoid is proposed. It is based upon solving Overdetermined Yule-Walker equations using constrained least-squares techniques. A Gauss-Newton algorithm is derived for proceeding to the constrained minimization. Simulation results prove the superiority of the new method over the unconstrained method, specially for a small number of equations.

1. INTRODUCTION

In many applications [1-4], it is known that a multiplicative model could be more precise than the usual additive noise model. This is the case, for example, when train speed is to be estimated from an on-board Doppler radar: the sum of phase and frequency shifted echoes (due to the reflections from different points of the track) results in a signal whose amplitude is slowly fluctuating [1]. In the spectral domain, instead of a single peak at the Doppler frequency, one can observe a large bandwith and several peaks around the Doppler frequency. Hence, it becomes difficult to accurately recover the frequency of interest. This phenomenom is also encountered in radar systems with slowly-fluctuating targets [2], in communications where the changing reflective characteristics of the channel or multipath trajects result in amplitude modulation [3], and in the propagation of acoustic signals through the ocean [4]. In a recent paper [1], we introduced a new model (called ARCOS): the signal was modelled as a sine wave (whose frequency, referred to as the *Doppler frequency*, is to be estimated) amplitude modulated by a lowpass autoregressive (AR) process.

Due to the ARMA-like form of the spectrum, the Doppler frequency was estimated in a two-step procedure. First, an ARMA model was fitted to the data (using the Overdetermined Yule-Walker (OYW) method). In a second step, a search for the centroid of the set of poles was performed. This estimator suffers from some problems:

1) it is an *indirect method* which does not make full use of the a priori knowledge of the particular form of the AR coefficients of the ARMA spectrum

2) the Doppler frequency estimation is coupled with that of the lowpass AR coefficients estimation and depends on the ARMA parameters estimates

3) we perform polynomial rooting which is time consuming.

In this paper, we propose a new method based upon constrained least-squares techniques. It proceeds to the resolution of the OYW equations by constraining the solution to have a particular form.

2. NOTATIONS

Consider the following ARCOS process:

$$x(n) = y(n).\cos(n\omega_0 + \varphi) \qquad (1)$$

Here:

- $y(n)$ is an autoregressive process of order p (AR(p)) defined by

$$y(n) = -\sum_{k=1}^{p} c_k . y(n-k) + u(n) \qquad (2)$$

and $\{u(n)\}$ are independent and identically distributed normal random variables $\aleph(0,\sigma^2)$.

- φ is a random phase, uniformly distributed on $(0,2\pi]$, independent of $y(n)$

Let $C(z) = \sum_{k=0}^{p} c_k . z^{-k}$ with $c_0 \overset{\Delta}{=} 1$. It is admitted that the ARCOS in equation (1) models signals with a slowly varying amplitude.

Under the previous hypotheses, it can be shown [1] that $x(n)$ is a zero mean stationary process, with

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autocorrelation function and power spectral density given by:

$$r_x(k) \overset{\Delta}{=} E\{x(n)x(n+k)\} = \frac{1}{2} r_y(k).\cos(k\omega_0) \quad (3)$$

$$S_x(z) = \frac{\lambda^2.B(z).B^*(1/z^*)}{A(z).A^*(1/z^*)} \quad (4)$$

where λ^2 and polynomial B(z), of degree p, depend on $\{c_1,\dots,c_p,\sigma^2,\omega_0\}$ and where

$$A(z) = \sum_{k=0}^{2p} a_k.z^{-k} = C(z.e^{j\omega_0}).C(z.e^{-j\omega_0}) \quad (5)$$

An ARCOS process is therefore *spectrally equivalent* to an ARMA(2p,p) with the following main property: ω_0 is the centroid of the set of (positive) angular frequencies of the zeroes of $A(z)$. For any generic vector $\underline{\theta} = (c_1,..,c_p,\omega_0)^T$, we let $\underline{a}(\underline{\theta}) = (a_1(\underline{\theta}),\dots,a_{2p}(\underline{\theta}))^T$ denote the vector that satisfies:

$$a_m(\underline{\theta}) = \sum_{k=0}^{p} c_k.c_{m-k}.\cos((2k-m)\omega_0), \quad m = 1..2p \quad (6)$$

with the convention that $c_k = 0 \quad \text{for } k \notin [0,p]$. By identifying the coefficients of z^{-m} in both sides of (5), it is readily seen that $\underline{a}(\underline{\theta})$ corresponds to the AR parameters of the ARMA-like spectrum of an ARCOS process where:

- $(c_1,\dots,c_p)$ are the p AR parameters of the lowpass AR(p) envelope
- ω_0 is the Doppler frequency.

As $\underline{a}(\underline{\theta})$ is a continuous and differentiable function, we define the Jacobian matrix as $D(\underline{\theta}) \overset{\Delta}{=} \frac{\partial \underline{a}^T(\underline{\theta})}{\partial \underline{\theta}}$.

3. FREQUENCY ESTIMATION

Given the ARMA(2p,p) form of the spectrum, the autocorrelation function(ACF) of x(n) satisfies the so-called Overdetermined Yule-Walker equations [5-8]:

$$R\underline{a} = -\underline{r} \quad (7)$$

where $\underline{r} = [r_x(p+1),\dots,r_x(p+M)]^T$ and R is a (M|2p) matrix whose elements are $R(i,j) = r_x(p+i-j), \ i = 1..M, j = 1..2p$. Here M>2p denotes the number of equations in (7). We note

$$\hat{r}_x(m) = \frac{1}{N-m} \sum_{n=1}^{N-m} x(n).x(n+m) \quad (8)$$

the usual unbiased estimator of the ACF, from a sample of N data points. Also, let $\hat{R}$ and $\hat{\underline{r}}$ denote consistent estimates of R and $\underline{r}$, formed by using the sample covariances (8).

The principle of the new method is to find a vector $\underline{a}$ that minimizes $\|\hat{R}\underline{a} + \hat{\underline{r}}\|^2$, subject to the constraint that it has the specific form $\underline{a}(\underline{\theta})$ of (6). In other terms, the ARCOS parameter vector $\underline{\theta}$ is estimated as the minimizer of the following criterion:

$$\hat{\underline{\theta}}^c = \arg\min_{\underline{\theta}} \left\{ J_c(\underline{\theta}) = \|\underline{\varepsilon}_c(\underline{\theta})\|^2 = \|\hat{R}.\underline{a}(\underline{\theta}) + \hat{\underline{r}}\|^2 \right\} \quad (9)$$

where $\|\underline{\xi}\|^2 \overset{\Delta}{=} \underline{\xi}^T \underline{\xi}$. This is a quadratic least-squares problem with a non-linear constraint. Alternatively, it can be viewed as a non-linear least-squares minimization problem. For this reason, $\hat{\underline{a}}^c \overset{\Delta}{=} \underline{a}(\hat{\underline{\theta}}^c)$ and $\hat{\underline{\theta}}^c$ are said to be **constrained estimates.** The minimization is performed using a Gauss-Newton technique, the expression for the (n+1)st iteration being:

$$\hat{\underline{\theta}}^{(n+1)} = \hat{\underline{\theta}}^{(n)} - \left[\left[\frac{\partial^2 J_c(\underline{\theta})}{\partial \underline{\theta}\, \partial \underline{\theta}^T} \right]^{-1} . \frac{\partial J_c(\underline{\theta})}{\partial \underline{\theta}} \right]_{\underline{\theta} = \hat{\underline{\theta}}^{(n)}} \quad (10)$$

Differentiation of the cost function gives the following expressions for the gradient and the Hessian:

$$\frac{\partial J_c(\underline{\theta})}{\partial \underline{\theta}} = 2 \frac{\partial \underline{\varepsilon}_c^T(\underline{\theta})}{\partial \underline{\theta}} . \underline{\varepsilon}_c \quad (11)$$

$$\frac{\partial^2 J_c(\underline{\theta})}{\partial \underline{\theta}\, \partial \underline{\theta}^T} = 2 \frac{\partial \underline{\varepsilon}_c^T(\underline{\theta})}{\partial \underline{\theta}} . \frac{\partial \underline{\varepsilon}_c(\underline{\theta})}{\partial \underline{\theta}^T} + \underline{\varepsilon}_c''(\underline{\theta}).\underline{\varepsilon}_c(\underline{\theta}) \quad (12)$$

The first term of the Hessian is positive definite for parameter vectors close to the true value. The second term, which is proportional to the error, tends to zero as the estimate approaches the true value and can be neglected. This is a standard approximation and it ensures that the Hessian remains positive definite. By noting first that:

$$\frac{\partial \underline{\varepsilon}_c^T(\underline{\theta})}{\partial \underline{\theta}} = \frac{\partial \underline{a}^T(\underline{\theta})}{\partial \underline{\theta}}.\hat{R}^T = D(\underline{\theta}).\hat{R}^T \tag{13}$$

$$\frac{\partial J_c(\underline{\theta})}{\partial \underline{\theta}} = 2\frac{\partial \underline{\varepsilon}_c^T(\underline{\theta})}{\partial \underline{\theta}}.\underline{\varepsilon}_c = 2.D(\underline{\theta}).\hat{R}^T.\underline{\varepsilon}_c(\underline{\theta}) \tag{14}$$

Inserting (11)-(14) in (10), the recursion becomes:

$$\hat{\underline{\theta}}^{(n+1)} = \hat{\underline{\theta}}^{(n)} - \left[\left[D(\underline{\theta}).\hat{R}^T.\hat{R}.D^T(\underline{\theta})\right]^{-1}.D(\underline{\theta}).\hat{R}^T.\underline{\varepsilon}_c(\underline{\theta})\right]_{\underline{\theta}=\hat{\underline{\theta}}^{(n)}} \tag{15}$$

It remains to derive $D(\underline{\theta})$ to complete the derivation of the minimization. From (6), it is easily shown that:

$$\frac{\partial a_j}{\partial c_i} = 2.c_{j-i}.\cos((2i-j)\omega_0)$$

$$\frac{\partial a_j}{\partial \omega_0} = -\sum_{k=0}^{p}(2k-j).c_k.c_{j-k}.\sin((2k-j)\omega_0) \tag{16}$$

where we have used the notation $\theta_i = c_i,\ i = 1..p$ and $\theta_{p+1} = \omega_0$. The previous equations (15)-(16) describe the minimization technique. The iterative procedure can be stopped whenever

$$\left\|\hat{\underline{\theta}}^{(n+1)} - \hat{\underline{\theta}}^{(n)}\right\| \le \delta \left\|\hat{\underline{\theta}}^{(n)}\right\| \tag{17}$$

where δ is a user-defined threshold, fixing the relative precision of the solution. The Doppler frequency estimate $\hat{\omega}_0^c \stackrel{\Delta}{=} \hat{\underline{\theta}}^c(p+1)$ is the (p+1)st element of $\hat{\underline{\theta}}^c$.

NB: It should be noted that the number of iterations needed for (15) to converge is small (typically 2 to 6 iterations are sufficient [9]). It mainly depends on δ and the initial vector $\hat{\underline{\theta}}^{(0)}$.

The advantage of the constrained method over the previous is twofold:

- it incorporates directly in the estimation of $\hat{\underline{a}}^c$ the a priori information on ARCOS processes, which should give more accuracy. More, this method gives a direct estimation of the frequency and of the lowpass AR parameters.
- the dimension of the vector to be estimated is decreased since we perform a minimization on a vector of dimension p+1 instead of 2p. More, we do not need to resort to polynomial rooting for estimating the Doppler frequency since this latter is implicitely given.

4. NUMERICAL EXAMPLES

In this section, we provide illustrations of the performances of the new method and compare it with the unconstrained method of [1]. We consider an ARCOS process defined by the following parameter vector $\underline{\theta}^T = \left[-2*0.95*\cos(2\pi*0.006), 0.95^2, 2\pi*0.18\right]$

500 Monte-Carlo simulations were run to estimate the variances of the estimates. The number of samples was varied between 256 and 4096. Figures 1,2,3 show the variances for the previous method (solid lines) and the new method (dashed lines) for different choices of the number of YW equations M in (9).

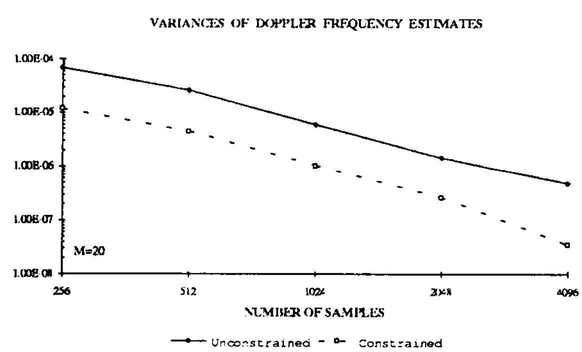

Figure 1: Empirical variances of constrained (dashed lines) and unconstrained estimates (solid lines). M=20

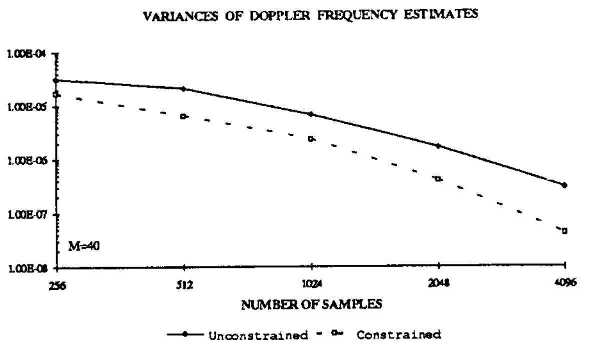

Figure 2: Empirical variances of constrained (dashed lines) and unconstrained estimates (solid lines). M=40

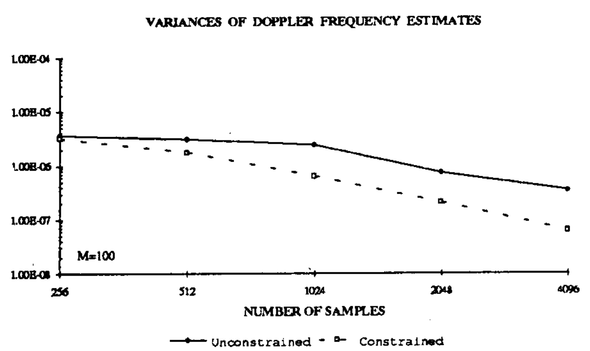

Figure 3: Empirical variances of constrained (dashed lines) and unconstrained estimates (solid lines). M=100

It is shown that **the constrained estimate always outperforms the unconstrained estimate.** The improvement is shown to vary between 0.6dB and 11dB on the variance. It is worth noting that the improvement is all the more important (between 7 and 11dB when M=20) as M is small. This means that it is preferable to use the constrained method when the number of equations M is small. The fact that the improvement increases while the number of equations decreases is a very interesting feature. Indeed, for a given level of performance (*ie* for a constant variance), the constrained method requires less covariance lags than the unconstrained one. As the main computational task is the estimation of the covariance sequence and since the minimization involves a smaller number of equations, this can result in significant savings from a computational point of view. It should also be noted that, for N large, the variance of the constrained estimate increases while M increases. This is in contrast with the unconstrained method.

5. CONCLUSIONS

In this paper, we proposed a new method for estimating the frequency of a random amplitude sinusoid. This method is based on constrained least-square techniques. The estimated ARMA parameter vector is constrained to belong to a subset defined by the very particular form of ARCOS spectrum. A Gauss-Newton algorithm is derived for proceeding to the constrained minimization. Simulation results prove the superiority of the new method over the unconstrained method. It is shown that the improvement is all the more important as the number of Yule-Walker equations is small.

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