STRUCTURED AUTOREGRESSIVE INSTANTANEOUS PHASE AND FREQUENCY ESTIMATION

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ABSTRACT

A new approach to estimate the phase and amplitude signal parameters of a quite general class of complex valued signals is presented. The proposed algorithm can estimate the signal parameters of a sum of complex signals, the amplitudes may be time varying and the phase functions are modelled by some continuous functions $a_l(t)$. The data can be evenly or unevenly sampled in time. The signal parameter estimates minimizes a loss function based on the prediction errors of a new, time dependent, structured autoregressive filter. The instantaneous phase and frequency is easily obtained from the estimated signal parameters. The structured AR filter is a model based time-frequency representation.

1. INTRODUCTION

The estimation of the instantaneous phase and frequency of complex nonlinear phase complex signals is an important and difficult task in a variety of applications. Examples are synthetic aperture radar where the desired image is blurred and distorted due to unexpected phase variations. The image can be focused if the phase variations are estimated and compensated for. Doppler radar signals from maneuvering targets have nonlinear phase. Estimation of the phase function coefficients can be used to obtain the radial acceleration (and higher order moments) without the use of Kalman filtering. Passive radar intelligence systems can use the estimated phase function parameters to identify the transmitting radar type. The phase functions in the examples above are generally nonlinear and may have a time varying amplitude. Also, the measurements may consist of a sum of signals of the aforementioned type. A quite general continuous time model for the signals under consideration is

$$x(t) = \sum_{l=1}^{p} b_l(t) e^{j\, a_l(t)} \tag{1}$$

where $b_l(t)$ and $a_l(t)$ are continuous real valued functions that model the time varying amplitude and phase, respectively, and t denotes time. The instantaneous phase and frequency of the l:th signal component at time t is defined as $a_l(t)$ and $\frac{d}{dt}a_l(t)$.

The classical approach to phase estimation is by phase locked loops, which is a non parametric approach. Maximum likelihood estimation could be used if a parametric model is assumed, but this approach is impractical due to extreme numerical difficulties. The special case of chirp signals (of constant amplitude) has been addressed in a number of papers, see for example [2] and the references therein. An approach to estimate and classify a single ($p = 1$) nonlinear phase complex signal of constant amplitude based on an integral transform was presented in [1]. Parametric estimation of multiple amplitude modulated nonlinear phase complex signals from noisy measurements, however, appear not to have been reported.

Here, a new approach to the signal parameter estimation of the quite general class of signals modeled by (1) is introduced. The signal parameter estimates are found from sampled data as the minimizer of a loss function based on the prediction errors of a new time dependent structured autoregressive (AR) filter. Unevenly sampled data, i.e. a time varying sampling interval, is allowed. The instantaneous phase and frequency is easily obtained from the estimated signal parameters.

The structured AR filter is a model based time-frequency representation of the signal and can, for example, be used to plot the time varying spectral density. The structured AR filter appears to be one of the first model-based time-frequency representations to be reported, c.f. [4].

2. PROBLEM SETUP

Consider a complex valued signal modeled by (1). The time continuous signal $x(t)$ is sampled at time instants $\{t_k\}_{k=1}^{N}$. The sampling interval $(t_k - t_{k-1})$ may vary. The measurements are corrupted by additive identically distributed independent random variables $e(t_k)$ with zero mean and variance σ^2:

$$y(t_k) = x(t_k) + e(t_k) \tag{2}$$

Let the functions $a_l(t)$ and $b_l(t)$ be parametrized by some parameter vector ϑ. For example, if the signal is a single high order polynomial phase signal with constant amplitude,

$$x(t_k) = b_0 e^{j \sum_{r=0}^{q} a_r t^r} \tag{3}$$

then $\vartheta^T = (a_1, ..., a_q, b_0)$.

The algorithm to be presented enables estimation of the continuous time signal parameters defined in ϑ.

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3. THE STRUCTURED AR FILTER

The prediction error of an n:th order AR-filter can be written as

$$\varepsilon(t_k) = y(t_k) + \sum_{d=1}^{n} c_d\, y(t_{k-d}) \qquad (4)$$

where $(c_1, \ldots, .c_n) \triangleq \theta^T$ denote the AR-parameters. It is well known how θ shall be choosen such that the variance of the prediction errors is minimized in the case of wide sence stationary (WSS) and correlation ergodic signals. If only one realisation is available and the signal is not WSS and correlation ergodic these AR parameters are of little value of obvious reasons. If, however, an *ensamble* of M realizations is available then θ can be calculated at time instant t_k from the ensemble as:

$$\begin{aligned} \hat{\theta}(t_k) &= -\hat{R}(t_k)^{-1}\hat{r}(t_k) && (5)\\ \hat{R}(t_k) &= \begin{pmatrix} \hat{\rho}(t_k;1,1) & \ldots & \hat{\rho}(t_k;1,n) \\ \vdots & \ddots & \vdots \\ \hat{\rho}(t_k;n,1) & \ldots & \hat{\rho}(t_k;n,n) \end{pmatrix} && (6)\\ \hat{r}^T(t_k) &= (\hat{\rho}(t_k;1,0)\ldots\hat{\rho}(t_k;n,0)) && (7)\\ \hat{\rho}(t_k;u,v) &= \frac{1}{M}\sum_{m=1}^{M} y_m^*(t_{k-u})y_m(t_{k-v}) && (8) \end{aligned}$$

where u and v are integers and $y_m(t_k)$ referres to a measurement from the m:th realization. The (ensemble) AR parameters (5) are time varying in the general case and can be estimated even if the sampling period $t_k - t_{k-1}$ is not constant.

Unfortunately (5) is of little value in practical situations since mostly only one realization is available. The *analytical* AR-filter of order n that minimizes the expected prediction error variance of the ensamble at time t_k can, however, easily be derived as a function of the signal parameters ϑ. Consider the (hypthetical) case when an infinite number of realizations are available, i.e. when $M \to \infty$. From (8), (2) and (1) it follows that the analytical covariance function at time t_k can be written as

$$\begin{aligned} &\rho(t_k;\vartheta,u,v) \triangleq \\ \triangleq\; & \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M} y_m^*(t_{k-u})y_m(t_{k-v}) \\ =\; & \Big(\sum_{l=1}^{p} b_l(t_{k-u})e^{-j\,a_l(t_{k-u})}\Big)\Big(\sum_{q=1}^{p} b_q(t_{k-v})e^{j\,a_q(t_{k-v})}\Big) \\ & +\sigma^2\delta_{u,v} \qquad (9) \end{aligned}$$

Introduce the notations

$$\begin{aligned} s(t_k;\vartheta,u) &\triangleq \sum_{l=1}^{p} b_l(t_{k-u})e^{j\,a_l(t_{k-u})} && (10)\\ \mathcal{S}(t_k;\vartheta) &\triangleq (s(t_k;\vartheta,1),\ldots,s(t_k;\vartheta,n)) && (11) \end{aligned}$$

Then the analytical (ensemble) covariance function, vector and matrix at time t_k can be written as

$$\begin{aligned} \rho(t_k;\vartheta,u,v) &= s^*(t_k;\vartheta,u)s(t_k;\vartheta,v) + \sigma^2\delta_{u,v} && (12)\\ r(t_k;\vartheta) &= \mathcal{S}^H(t_k;\vartheta)s(t_k;\vartheta,0) && (13)\\ R(t_k;\vartheta) &= \mathcal{S}^H(t_k;\vartheta)\mathcal{S}(t_k;\vartheta) + \sigma^2 I_n && (14) \end{aligned}$$

where I_n denotes an $(n \mid n)$ identity matrix. From (12)-(14) the analytical AR parameters at time t_k can easily be calculated as

$$\theta(t_k;\vartheta) = -R(t_k;\vartheta)^{-1}r(t_k;\vartheta) \qquad (15)$$

Formula (15) involves an inversion of the $(n \mid n)$ matrix $R(t_k;\vartheta)$ which implies heavy computations. This can be avoided by a strightforward use of the matrix inversion lemma; $(A+BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$. Using $A = \sigma^2 I_n$, $B = \mathcal{S}^H(t_k;\vartheta)$, $C = 1$ and $D = \mathcal{S}(t_k;\vartheta)$ and assuming that $\sigma^2 \neq 0$ gives the final formula for the structured AR-filter:

$$\begin{aligned} \theta(t_k;\vartheta) &\triangleq \frac{1}{\sigma^2}\Big\{\frac{1}{\mathcal{S}(t_k;\vartheta)\mathcal{S}^H(t_k;\vartheta)+\sigma^2} \\ &\mathcal{S}^H(t_k;\vartheta)\mathcal{S}(t_k;\vartheta) - I_n\Big\}\mathcal{S}^H(t_k;\vartheta)s(t_k,0;\vartheta) \qquad (16) \end{aligned}$$

An intuitive understanding for the structured AR-filter is gained by studying how the structured AR-parameters use previous measurements to make a prediction. The measurements $\{y(t_{k-u})\}_{u=1}^n$ are rotated by the complex valued structured AR-parameters $\{c_u(t_k;\vartheta)\}_{u=1}^n$ such that they are aligned with $y(t_k)$, i.e. $y(t_{k-u})$ is rotated $a(t_k) - a(t_{k-u})$ radians. The magnitude of the structured AR parameters are, by construction, such that the expected (ensemble) prediction error is minimized. Indeed, the structured AR-filter can be derived using that the argument of the u:th structured AR-parameters must equal $a(t_k) - a(t_{k-u})$ since the noise $e(t_k)$ is uncorrelated with the signal $x(t_k)$. Then an optimazation of the magnitude of the structured AR parameters such that the expected (ensemble) prediction variance becomes minimal gives the same result as in (16).

4. SIGNAL PARAMETER ESTIMATION

The estimate of the signal parameters ϑ is defined as

$$\hat{\vartheta} = \arg \min_{\vartheta} V_N(\vartheta) \qquad (17)$$

where the criterion function $V_N(\vartheta)$ is given by

$$V_N(\vartheta) = \frac{1}{N}\sum_{k=1}^{N} \epsilon^2(t_k;\vartheta) \qquad (18)$$

Several choises of the loss function $V_N(\vartheta)$ exist but are not discussed here.

It follows from the construction of $\theta(t_k;\vartheta)$ that ϑ_0 minimizes $E[V_N(\vartheta)]$, where ϑ_0 denotes the true signal parameters. Straightforward manipulation gives

$$\begin{aligned} \mathrm{E}[V_N(\vartheta)] &= \mathrm{E}[\frac{1}{2N}\sum_{k=1}^{N}\epsilon^2(t_k;\vartheta)] = \frac{1}{2N}\sum_{k=1}^{N}\mathrm{E}[\epsilon^2(t_k;\vartheta)] \\ &\geq \frac{1}{2N}\sum_{k=1}^{N}\mathrm{E}[\epsilon^2(t_k;\vartheta_0)] = \mathrm{E}[V_N(\vartheta_0)] \qquad (19) \end{aligned}$$

Note that (19) holds independently of the structured AR filter length, the number of data N and even if the data has been sampled unevenly in time.

The minimazation of (17) is easily implemented using e.g. a Gauss Newton search. The derivatives of $\theta(t_k;\vartheta)$ with respect to the signal parameters in ϑ are straightforward to derive. The search can be implemented off- or on-line (tracking).

Unfortunately the loss function $V_N(\vartheta)$ contains local minima, so care must be taken such that the search converges to the global minima. The loss function has, however, some nice properties that have been confirmed by numerical studies. The radius of attraction of the global minima increases with decreasing structured AR order n. In fact, in the case of a "pure" complex signal where $a_l(t) = \phi_l + w_l t$ there are no local minima if $n = 1$. In the case of non stationary signals the radius of attraction of the global minima increases with decreasing number of data, N. Hence a low order structured AR filter length can be used on a small portion of the data in order to get initial estimates. Then the data set and the structured AR length can be increased in order to get more accurate estimates.

Estimates of the instantaneous phase and frequency are easily obtained as $\hat{a}_l(t)$ and $\frac{d}{dt}\hat{a}_l(t)$ respectively.

5. TIME-FREQUENCY REPRESENTATION

The structured AR filter is a time-frequency representation (TFR). Once the signal parameters $\hat{\vartheta}$ have been estimated it is a simple matter to get an estimate of the instantaneous power spectrum density (PSD) as follows. Given $\hat{\vartheta}$ choose a constant "sampling" interval $\Delta = t_k - t_{k-1}$ and a structured AR model length n. Then calculate the structured AR parameters from (10)- (14) and (16) and substitute into the theoretical PSD to obtain

$$\mathrm{P_{AR}}(t_k,\omega) = \left\| \frac{1}{1+\sum_{d=1}^{n} c_d(t_k;\hat{\vartheta})e^{-jd\omega}} \right\| \qquad (20)$$

where $\omega \in [0, 2\pi)$ is a discrete frequency.

The design parameters n and Δ can freely be chosen by the user to suit the purposes. In a sense the structured AR filter can be viewed as an instrument to get the instantaneous PSD from the signal parameters $\hat{\vartheta}$. Note that $\mathrm{P_{AR}}(t_k,\omega)$ at time t_k is derived from *all* available data $\{y(t_k)\}_{k=1}^{N}$. This is not the case in, for example, the short time Fourier transform (STFT) where only a small portion of the data from a sliding window is used to get an estimate of the instantaneous PSD.

There is a trade off between time and frequency resolution determined by the choise of instantaneous structured AR "observation window" width $(t_k - t_{k-n}) = n\Delta$. The "observation window" can be viewed as a sliding window length similar to that in the STFT. Using a small "observation window" will yield good time resolution at the cost of poor frequency resolution and vice versa. The fit of the structured AR model improves with increasing model order n. A good model fit implies a good spectral resolution. Hence n shall be chosen large enough such that a good spectral resolution is achieved, but not too large such that a satisfactory time resolution is maintained.

6. NUMERICAL STUDY

The structured AR estimation method has been studied by means of Monte Carlo simulation. For each set of parameters 50 independent experiments were run.

6.1. Dependence on SNR

Consider the same signal as that used in the example in [1], i.e. a single complex signal with constant amplitude and polynomial phase of order 3; $a(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, $\{a_i\}_{i=0}^{3} = \{1, 200\pi, 96\pi, 2048\pi\}$. The signal was evenly sampled in the interval [0,500/2048] and $N = 100$. This is only a fraction of the data used in [1] where $N = 2048$.

The minima of $V_N(\hat{\vartheta})$ w.r.t. the time invariant amplitude is very flat in the case of a single complex signal. Hence the search for $\hat{\vartheta}$ becomes ill conditioned if b_0 is included, so b_0 is preferrably fixed to a guessed value during the search. Therefore, and since the structured AR filter is independent of the initial phase a_0 for the case of a single complex signal, only $\{a_i\}_{i=1}^{3}$ were estimated using the structured AR method and then $\{a_0, b_0\}$ were estimated using a least squares fit. The value of (the fixed) b_0 used in the search was set to to a random value such that the $\mathrm{SNR} \triangleq \frac{b_0^2}{2\sigma^2}$ was uniformly distributed in $[\mathrm{SNR}_0 - 5\mathrm{dB}, \mathrm{SNR}_0 + 5\mathrm{dB}]$, where SNR_0 denotes the true SNR.

Figure 1 shows the relative efficiency (the mean square error divided by the Cramer-Rao lower bound where the Cramer-Rao lower bound was derived in [3]) for $\hat{a}_3$ when $n = 8$ and 16, respectively. The behaviour of the relative

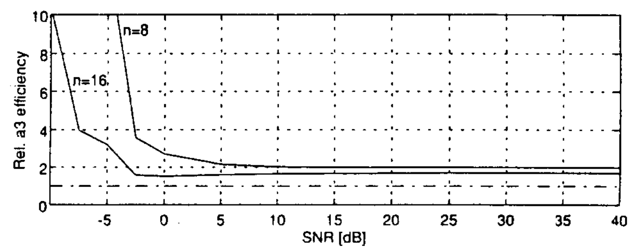

Figure 1: Relative efficiency versus SNR, $n = 8, 16$.

efficiency of $\{\hat{a}_i\}_{i=0}^{2}$ is very similar to that illustrated in Fig 1 and therefore not shown here. The relative efficiency of $\{\hat{a}_i\}_{i=0}^{3}$ at SNR=0dB for $n = 16$ is {2.5, 2.3, 1.8, 1.5} respectively, a result which is comparable to 1.5-1.7 that was reported in [1]. In other similar examples a relative efficiency of 1.2 has been achieved when using the structured AR algorithm.

The performance of the algorithm proposed in [1] deteriorates rapidly for SNR below a threshold. This is also the case for the structured AR algorithm. The threshold is, however, approximately 4dB ($n = 8$) and 8dB ($n = 16$) *lower* for the structured AR algorithm than for that presented in [1]. The threshold decreases with increasing n. The dependency of the threshold on n is under current investigation.

6.2. Dependence on n

Figure 2 shows the relative efficiency of $\hat{a}_3$ versus n for fixed SNR=10dB for the same case as in Fig 1. The corresponding curves for $\{\hat{a}_i\}_{i=0}^{2}$ are similar and therefore omitted. For small n, the performance improves with increasing n until an optimal value is reached. Then, as n is increased beyond the optimal value, the performance deteriorates. The optimal choise of n has been found to depend on the number of data and SNR. The dependency is under current investigation.

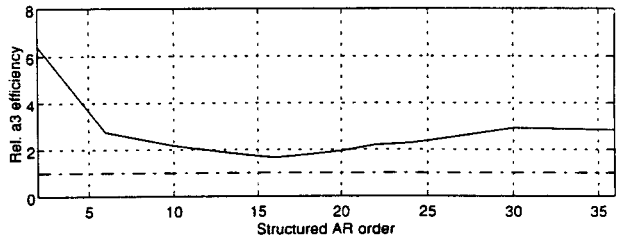

Figure 2: Relative efficiency versus n, SNR=10dB

6.3. Unevenly sampled data

The structured AR algorithm allowes unevenly sampled data in time. A numerical investigation has shown that the statistical properties of the estimates depend on the sampling strategy. Indeed, it is possible to achieve improved accuracy if the data is unevenly sampled in time. To illustrate, consider the same example as in Fig 2 but with data unevenly sampled. The data was sampled in four "bursts", each consisting of 25 samples (evenly) separated $\frac{1}{4096}$s apart. The "bursts" were in turn evenly spread in the same time interval as that used in the example corresponding to Fig 2. The relative improvement of the MSE, i.e. $\frac{MSE_1 - MSE_2}{MSE_1}$ where MSE_1 denotes the MSE in Fig 2 and MSE_2 denotes the MSE achieved when using unevenly sampled data, was 82%, 78%, 66% and 60% for $\{a_i\}_{i=0}^{3}$, respectively.

6.4. Time frequency representation

Figure 3 shows the spectogram derived using the STFT on a constant amplitude complex signal with polynomial phase of order 3; $\{a_i\}_{i=0}^{3} = \{1, 120\pi, -100\pi, 60\pi\}$. The signal was (evenly) sampled in the time interval [0,1]; $\Delta = 0.01$ and $N = 100$. Figure 4 shows the corresponding spectogram derived using the structured AR approach (20). The same "observation window" and "sampling interval" was used in both cases, i.e. the sliding window in the STFT was as wide as the structured AR filter was long and Δ was chosen to the sampling interval of the data. The "observation window" length (n) was chosen to 5.

The resolution when using $P_{AR}(t_k;\vartheta)$ is significantly better than that achieved when using the STFT. The spectogram from $P_{AR}(t_k;\vartheta)$ is not as sensitive to noise as that achieved when using the STFT, i.e. it is smoother.

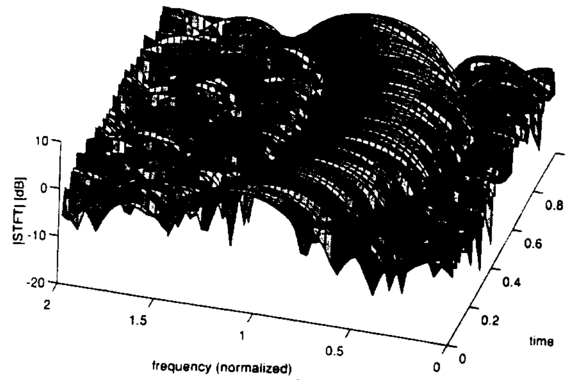

Figure 3: The short time Fourier transform

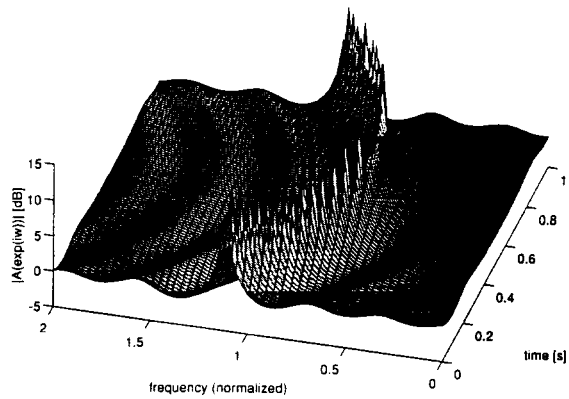

Figure 4: $P_{AR}(t_k;\vartheta)$

7. CONCLUDING REMARKS

The structured AR algorithm has been applied to a variety of signals such as sums of complex signals and signals with time varying amplitudes. The results confirm that the structured AR algorithm successfully can estimate the parameters of such signals. The results have, however, not been included here due to lack of space. A corresponding algorithm for real valued sinusoidal signals has been derived and verified as well.

The statistical properties of the structured AR algorithm and it's relative performance to other estimators and time-frequency representations is under current investigation.

8. REFERENCES

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