A PARAMETRIC MODEL FOR THE QUADRATIC SAMPLING OF A BANDLIMITED SIGNAL

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ABSTRACT

This paper treats the problem of ambiguity resolution using non-uniform sampling. This problem occurs for Doppler estimation in coherent pulsed Doppler radar. In this paper, we study the case where the duration between two samples is a linear function of time: quadratic sampling. Assuming that the continuous signal is stationary, the sampled signal will be non-stationary. The autocorrelation of this signal is derived and the Wigner distribution of the sampled signal related to the spectrum of the continuous signal. As a consequence, a time frequency relief of the signal will verify symmetries. These constraints, assuming an AR evolutive model for the sampled signal and band limitation for the continuous signal, allow the derivation of a particular time varying model for the samples. An associated estimation algorithm, leading to the unfolded spectrum is then proposed.

1. Motivation of this work

In a coherent pulsed Doppler radar, [1], the echo distribution in Doppler is retrieved from the power spectrum of a signal sampled at the pulse emission instants t_n. Consequently, if the pulse repetition frequency (PRF) is constant, *i.e.* $t_n = n/PRF$, this distribution can only be estimated modulo PRF (*e.g.*, for an ideal point target we obtain after coherent demodulation and pulse integration $A\exp(j2\pi(f_D/PRF)n)$). In many systems requiring low PRF to avoid range ambiguity, as surveillance radar, this ambiguity occurs.

To overcome this aliasing effect a solution is to emit pulses at nonuniform intervals. A widely employed technique is the use of multiple PRF, the ambiguity resolution is achieved by searching for coincidence between unfolded Doppler estimates for each PRF, [2]. The main two lacks of this solution are that it does not apply to a wide band process and that the signal coherence during the whole observation time is not fully exploited.

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To overcome these problems, we propose the use of a quadratic pulse emission law, *i.e.* the duration between two pulses is a linear function of time.

This communication is organized as follows: the first section analyses the effects of the quadratic sampling on a continuous stationary signal. In the second section, under band limited assumption, a non stationary autoregressive model is derived for the sampled signal. Section three presents an estimation algorithm for this model, leading to the unfolded spectrum (AR parameters + ambiguity order). In the fourth section, performances of the proposed method are evaluated by computer simulations in the lines spectrum case.

2. Properties of a quadratically sampled signal

We will consider that:

$$t_n = (\alpha/2)n^2 + n, \quad \alpha > 0,$$

and the sample $x(t_n)$ will be noted x_n.

To enlight the ambiguity resolution principle, consider the elementary case where the continuous time signal $x(t)$ is a sine wave with frequency f_D. As a result x_n is a chirp with sweep rate αf_D, quantity that can be theoretically estimated without ambiguity.

To derive a more detailed analyzis of the effect of the sampling we will assume that $x(t)$ is harmonizable and stationary, *i.e.*, [3]:

$$x(t) = \int_{-\infty}^{+\infty} \exp(j2\pi ft)dX(f)$$

$$E\{dX(f)dX^*(f')\} = \delta(f-f')dS(f)df'.$$

Substituting t by t_n in this expression leads to the following representation for the discrete time signal x_n:

$$x_n = \int_{-\infty}^{+\infty} \exp(j2\pi((f\alpha/2)n^2 + fn))dX(f), \quad (1)$$

$$E\{dX(f)dX^*(f')\} = \delta(f-f')dS(f)df'. \quad (2)$$

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Now, contrarily to the uniformly sampled case ($\alpha = 0$), the chirps $\exp(j2\pi((f\alpha/2)n^2 + fn))$ are discernable for every value of f, hence, the integration domain in (1) is not reduced.

We can also notice that the representation (1) is no more doubly orthogonal: whereas the spectral increments are still orthogonals, the decompositions functions are not orthogonal. This result is natural, the signal x_n being obviously nonstationary.

Representation (1,2) allows the computation of the autocorrelation of x_n. Under sufficient conditions, we have:

$$\begin{aligned} E\{x_{n+k}x^*_{n-k}\} &= \int_{-\infty}^{+\infty} \exp(j4\pi f(\alpha n+1)k)dS(f) \\ &= 1/(\alpha n+1)\int_{-\infty}^{+\infty} \exp(j4\pi fk)dS(f/(\alpha n+1)). \end{aligned}$$

This equality can be inverted and leads to:

$$\begin{aligned} W(n,f) &= 2\sum_{k=-\infty}^{k=+\infty} E\{x_{n+k}x^*_{n-k}\}\exp(-j4\pi fk) \\ &= \frac{1}{\alpha n+1}\sum_{q=-\infty}^{q=+\infty} S(\frac{f+q/2}{\alpha n+1}) \\ W(n,f) &= S_{\alpha n+1}(f/(\alpha n+1)), \end{aligned} \quad (3)$$

where:

- $W(n,f)$ is the Wigner Ville distribution of x_n, [4],
- $S_{\alpha n+1}(f)$ is the spectrum of the signal $x(t)$ sampled at frequency $1/(\alpha n+1)$.

Consequently, the Wigner Ville distribution at instant n of the quadratically sampled signal equals the spectrum of the uniformly sampled signal at a sampling frequency $1/(\alpha n+1)$ and dilated by a factor $\alpha n + 1$.

3. Model derivation for the sampled signal

Our next purpose will be to derive a general model for x_n. This will be first achieved using an elementary reasoning on a quadratically sampled harmonic of the signal. The obtained result will then be recovered using Eqn. 3.

For this consider the sampling effect on the harmonic component of $x(t)$ at frequency $f_D = f_r + n_r$ where $n_r \in \mathbb{Z}$ is the ambiguity order and $f_r \in [0,1[$ the decimal part of f_D. The "instantaneous" frequency associated to this sampled component becomes:

$$\begin{aligned} f_i(n) &= \alpha n f_D + f_D \\ &\overset{(mod\ 1)}{=} \alpha n(f_r + n_r) + f_r \quad (4) \\ &= \alpha n_r n + (\alpha n+1)f_i(0), \quad (5) \end{aligned}$$

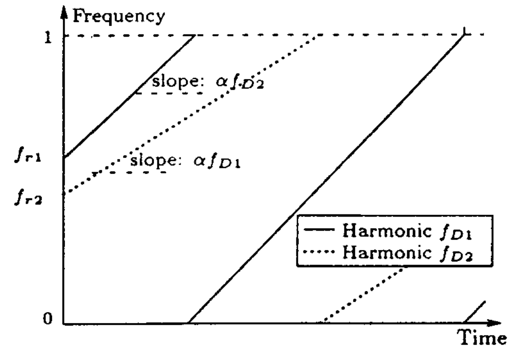

Figure 1: Time-frequency representation for a quadratically sampled bicomponent signal.

where $f_r = f_i(0)$. Fig. 1 sketches this result using a time-frequency representation of x_n, denoted $\rho(n,f)$. From this result, if we make the assumption that all the frequency components of the signal have the same ambiguity order n_r, *i.e.* the bandwith of the signal is smaller than 1, $\rho(n,f)$ will verify the fundamental relation:

$$\rho(n,f) = \rho\left(0, \frac{f-\alpha n_r n}{\alpha n+1}\right) \quad (6)$$

This relation can be directly obtained using Eqn. 3. In fact, under maximum unit bandwidth assumption, as Fig. 2 shows, the spectra of $x(t)$ sampled at a frequency 1 and $1/(\alpha n+1)$ are related by:

$$S_{\alpha n+1}(f) = \frac{1}{\alpha n+1}S_1(f - \alpha n_r n/(\alpha n+1)). \quad (7)$$

Substitution of this result in Eqn. 3 leads to:

$$\begin{aligned} W(n,f) &= S_{\alpha n+1}(f/(\alpha n+1)) \\ &= \frac{1}{\alpha n+1}S_1(f/(\alpha n+1) - \alpha n_r n/(\alpha n+1)) \\ W(n,f) &= \frac{1}{\alpha n+1}W(0,(f-\alpha n_r n)/(\alpha n+1)). \end{aligned}$$

This expression differs from Eqn. 6 by $1/(\alpha n+1)$. However, has it will be seen in the simulations, $\alpha n \ll 1$. Consequently, this term will be neglected in the following.

The model of x_n will necessary verify this condition. We guess for it in the sequel an evolutive AR model, [5]:

$$\rho(n,f) = \frac{\sigma^2(n)}{\left|\sum_{k=0}^{p} a_k(n)\exp(-2i\pi fk)\right|^2}, \quad (8)$$
$$a_0(n) = 1.$$

The point is now to establish the conditions verified by the parameters $\{a_k(n)\}_{k=1,p}$ and $\sigma^2(n)$ in order that

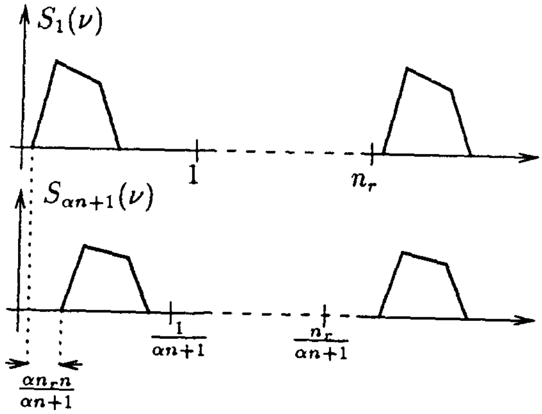

Figure 2: Spectrum of a bandlimited signal sampled at a frequency of 1 and $1/(\alpha n+1)$.

$\rho(n,f)$ given by Eqn. 8 verifies Eqn. 6. The system obtained replacing Eqn. 8 in Eqn. 6 is naturally non-linear and does not necessarily have a solution.

To alleviate these difficulties, we first suppress the time dependence on the white noise variance $\sigma^2(n)$ relaxing the constraint $a_0(n)=1$ for $n>0$. To cope with the remaining non-linearity issued from the squared absolute value, we consider a set of N frequencies, $\{f_k\}_{k=1,N}$ and impose the desired relation without the square absolute value, on this set. It is important to remark that this is a sufficient condition to obtain the desired relation on this set.

Under these hypothesis, the relation between parameters $\{a_k(n)\}_{k=0,p}$ and $\{a_k(0)\}_{k=1,p}$ becomes linear:

$$M(f_1,\ldots,f_N).\underline{a}(n)=M'(f_1,\ldots,f_N,\alpha,n_r).\underline{a}(0), \quad (9)$$

with:

$$\underline{a}(n)=(a_0(n),a_1(n),\ldots,a_p(n))^t,$$
$$\underline{a}(0)=(1,a_1(0),\ldots,a_p(0))^t.$$

We can finally demonstrate that the least squares solution of Eqn. 9 with $f_k=(k-1)/N$, $k=0\ldots N-1$, [6], is:

$$\underline{a}(n)=\Gamma(n,n_r,\alpha,N)\underline{a}(0), \quad (10)$$

where the $(k+1,q+1)$ element of the $p+1\times p+1$ matrix $\Gamma(n,n_r,\alpha,N)$ equals:

$$\gamma_{0,0}(n)=1 \quad (11)$$
$$\gamma_{k,q}(n)=\exp\{2i\pi\frac{\alpha n_r n}{\alpha n+1}q\}$$
$$\frac{\exp\{-2i\pi q/(\alpha n+1)\}-1}{\exp\{2i\pi(k-q/(\alpha n+1))/N\}-1}. \quad (12)$$

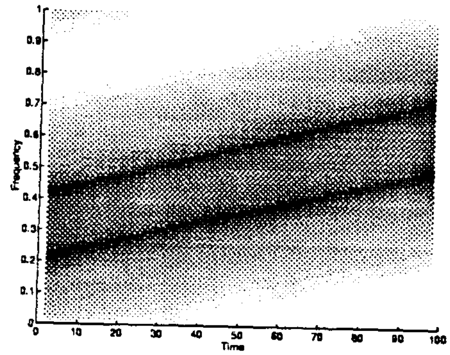

Figure 3: $\rho(n,f)$ for $\underline{a}(0)$ corresponding to two frequencies 0.2 and 0.4 with $n_r=6$.

To verify the validity of this model, *i.e.* Eqns. (10-12), a vector parameter $\underline{a}(0)$ and an ambiguity order n_r have been chosen. The sequence $\underline{a}(n)$ has been generated and $\rho(n,f)$ represented with gray levels in Fig. 3. A detailed analyze of this image shows that it verifies the symmetry properties required by Eqn. 6.

4. Estimation algorithm

The temporal regression associated to the evolutive AR model is:

$$a_0(n)x_n=-a_1(n)x_{n-1}-a_2(n)x_{n-2}-\ldots$$
$$-a_p(n)x_{n-p}+\varepsilon_n, \quad (13)$$
$$a_k(n)=\sum_{q=0}^{p}\gamma_{k,q}(n).a_q(0), \quad (14)$$
$$a_0(0)=1, E\{\varepsilon_n^2\}=\sigma^2. \quad (15)$$

If we replace in Eqn. 13 the AR coefficients by their expression and if we define the quantities:

$$z_q(n)=\sum_{k=0}^{p}\gamma_{k,q}(n)x_{n-k}, \quad (16)$$

we obtain the following temporal regression:

$$z_0(n)=-z_1(n)a_1(0)-\cdots-z_p(n)a_p(0)+\varepsilon_n$$
$$E\{\varepsilon_n^2\}=\sigma^2.$$

This result allows the estimation of the parameters $\{a_k(0)\}_{k=1,p}$ using a classical least squares algorithm[1]. However, this development requires the knowledge of the ambiguity order n_r to compute the $\gamma_{k,q}(n)$. We propose to solve this problem evaluating the $\{a_k(0)\}_{k=1,p}$

[1] In the case where $\alpha n \not\ll 1$, $z_q(n)$ must be replaced by $(\alpha n+1)z_q(n)$ in Eqn. 16

for a predefined set of n_r. This scanning can be performed with a reduced computational load observing that in Eqn. 12 n_r is present only in the first term. For each n_r the associated $\{a_k(0)\}_{k=1,p}$ are estimated and the variance of the corresponding sequence ε_n, generated by Eqn. 17, is calculated. The estimated value of n_r is naturally the one that minimizes σ^2.

5. Computer simulations and discussion

This algorithm has been validated using various computer simulations. To illustrate its behavior we consider the following two experiments:

1. $x(t)$ consists of three noisy sine waves at frequencies 6.12, 6.3 and 6.4. 128 samples of the associated nonuniformly sampled signal have been generated and corrupted by six realizations of a white noise with a SNR of 10dB. The algorithm parameters are $\alpha = 0.0003$, $N = 100$, $p = 9$.

2. $x(t)$ consists of two noisy sine waves at frequencies 5.11 and 5.22. 30 samples of the associated nonuniformly sampled signal have been generated and corrupted by six realizations of a white noise with a SNR of 20dB. The algorithm parameters are $\alpha = 0.0003$, $N = 100$, $p = 7$.

Fig. (4,5) show that the ambiguity order has been correctly estimated for each noise realization and represent the corresponding estimated spectrum.

We can notice that in the above simulations, the signal is composed by spectral lines. Whereas the proposed algorithm is not constraint to this case it is important to envisage it. In fact, the problem simplifies to the sweep rate estimation of chirps for wich the instantaneous frequency is given by Eqn. 4. Hence, many of the existing algorithm can be adapted to solve the problem: *e.g.*, the Wigner-Hough transform, [7], becomes an integration of the signal Wigner Ville distribution on lines defined by Eqn. 4. This conduces to the following spectral estimator:

$$\hat{S}(f) = \sum_n \sum_k x_{n+k} x^*_{n-k} \exp(-j4\pi k f(\alpha n + 1))$$

This estimator has been tested in our context and it appears that although it has the advantage that it does not require the band-limited hypothesis it has poor resolution compared to the proposed solution.

6. References

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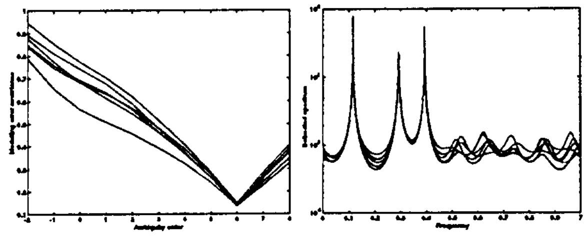

Figure 4: Estimated σ^2 function of n_r and estimated AR spectrum for $n_r = 6$, 128 samples, SNR=10dB.

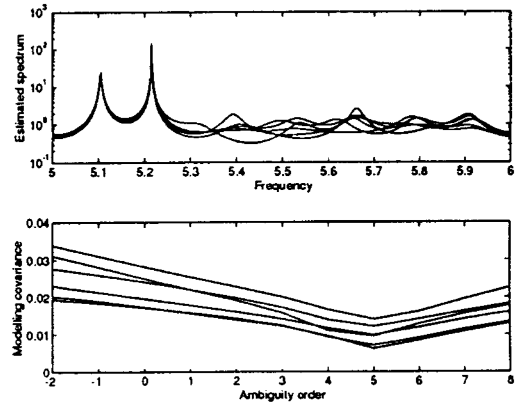

Figure 5: Estimated σ^2 function of n_r and estimated AR spectrum for $n_r = 5$, 30 samples, SNR=20dB.

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