STATISTICAL STUDY OF REFLECTION COEFFICIENTS DERIVED FROM A RANDOM AR PROCESS

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ABSTRACT

Being linked by non-linear relations, AR parameters and reflection coefficients cannot both be gaussian. However, a statistical study can show that these two sets of parameters are gaussian asymptotically. The aim of this paper is to show that the convergence rate of reflection coefficient distribution to the gaussian one depends on the position of AR model poles in the unit circle. An analysis of the reflection coefficient Taylor expansion around AR parameters is proposed to determine this convergence rate.

1. INTRODUCTION

Reflection Coefficients have successfully been used in Signal Processing. They have particularly desirable properties for quantization and coding. These coefficients can be computed from AutoRegressive (AR) parameters. There are at least three classes of applications for which AR parameters are assumed to be gaussian :

- in Estimation theory, AR parameters are deterministic and have to be estimated. According to the Mann and Wald theorem, most commonly used AR parameter estimators can be assumed to be gaussian when the parameters are estimated for sufficiently large data records [1].
- in Pattern Recognition, AR parameters are random and their statistics, which characterises intra-class scattering, is usually assumed to be known (generally gaussian).
- In the theory of random coefficient AR models, many methods have been developed using the gaussian assumption[6].

Consequently, in the sequel, the parameter vector of an AR process will be considered as a random gaussian variable denoted by a with mean $m_a = E(a)$ and covariance matrix $C_a = \sigma^2 M$, M being a constant matrix. In this case, a recursive way of determining the exact probabiliy density function (p.d.f.) of reflection coefficients from that of AR parameters can be derived [2][3]. This study is briefly recalled in the first part of the paper. In the previous applications, reflection coefficient p.d.f. tends to the gaussian one when $\sigma^2 \to 0$ [1]. The aim of this paper is to show that this tendency depends on the position of AR poles in the unit circle and this dependency is theoretically explained.

2. REFLECTION COEFFICIENT P.D.F.

Reflection coefficients are linked to AR parameters with the following relations :

$$\begin{array}{ll} 1 \le j \le i-1 & a_j^{(i-1)} = \dfrac{a_j^{(i)} - a_i^{(i)} a_{i-j}^{(i)}}{1 - \left[a_i^{(i)}\right]^2} \\ i = 1, ..., p & k_i = a_i^{(i)} \end{array} \qquad (1)$$

The parameters $a_j^{(i)}, j = 1, ..., i$ are the ith order linear predictor coefficients. For $i = p$, these parameters are identical to AR parameters :

$$j = 1, ..., p \qquad a_j^{(p)} = a_j$$

Vectors $\left[a_1^{(i)}, ..., a_i^{(i)}\right]$ are computed recursively for $i = p-1, ..., 1$ from AR parameters and relation (1). Each step allows us to determine one reflection coefficient $k_i = a_i^{(i)}$. The aim of this first part is to remind the reader of a recursive way of determining reflection coefficient p.d.f. as a function of that of AR parameters. For this, let us denote

$$V_i^T = \left[a_p^{(p)}, a_{p-1}^{(p-1)}, ..., a_i^{(i)}, a_{i-1}^{(i)}, a_{i-2}^{(i)}, ..., a_1^{(i)}\right]$$

This vector can be split into two parts :

- the first one has $p - i + 1$ components equal to the $p - i + 1$ reflection coefficients:

$$\left[a_p^{(p)}, a_{p-1}^{(p-1)}, ..., a_i^{(i)}\right] = [k_p, ..., k_i]$$

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- the second one has $i-1$ components:

$$\left[a_{i-1}^{(i)}, a_{i-2}^{(i)}, ..., a_1^{(i)}\right]$$

In particular, for $i = p$, we get the AR parameter vector V_p, the p.d.f. of which is assumed to be known and, for $i = 1$, we get the reflection coefficient vector V_1, the p.d.f. of which is under study.

The $p-i+1$ first components of vectors V_i and V_{i-1} are equal and the $i-1$ last ones are linked by relations (1) which can be inverted according to the parity of i. This allows us to determine the jacobian of the one-to-one transformation between V_i and V_{i-1} components (for more details see [3]). If $f_i(x_p, x_{p-1,...,}x_1)$denotes the V_i p.d.f., that of V_{i-1} can then be computed with the following relations:

- i odd

$$f_{i-1}(x_p, ..., x_1) = (1-x_i^2)^{(i-1)/2} f_i\left(x'\right) \quad (2)$$

with $x' = (x_p, ..., x_i, x_{i-1} + x_i x_1, ..., x_{1+} x_i x_{i-1})$.

- i even

$$f_{i-1}(x_p, ..., x_1) = (1+x_i)(1-x_i^2)^{(i-2)/2} f_i\left(x''\right) \quad (3)$$

with $x'' = (x_p, ..., x_i, x_{i-1} + x_i x_1, ..., (1+x_i)x_{i/2}, ..., x_{1+} x_i x_{i-1})$.

By means of p iterations, using (2) and (3), the reflection coefficient vector p.d.f. can be computed from that of the AR parameter vector.

3. REFLECTION COEFFICIENT GAUSSIANITY

From reflection coefficient p.d.f. computed in the first part of the paper, usual "distances" between probability density functions can be used to determine the "closeness" between reflection coefficient p.d.f. and the gaussian one [4]. For instance, the Kullback divergence between two p.d.f. $p_1$and p_2, which is defined by:

$$d_K(p_1, p_2) = \int_{\mathbf{R}} [p_2(x) - p_1(x)] \ln \frac{p_2(x)}{p_1(x)} dx$$

is often considered to determine how far away from each other two probability laws are. But relations between this distance and the AR model poles or parameters cannot be easily interpreted. In order to study the tendency of reflection coefficient p.d.f. to the gaussian one when parameter σ^2 tends to zero, a new approach is proposed. Using the same method as in [1], for low σ^2 values, a will only exhibit values close to m_a. Hence, $k = \Phi(a)$ will be close to $m_k = \Phi(m_a)$. A Taylor expansion then leads to the following relation :

$$k = \Phi(a) = \Phi(m_a) + \left.\frac{\partial\Phi(\alpha)}{\partial\alpha}\right|_{\alpha=m_a} (a-m_a) + \frac{1}{2}\left.\frac{\partial^2\Phi(\alpha)}{\partial\alpha^2}\right|_{\alpha=m_a} (a-m_a, a-m_a) + ...$$

$\frac{\partial\Phi(\alpha)}{\partial\alpha}$ and $\frac{\partial^2\Phi(\alpha)}{\partial\alpha^2}$ being the first and second order derivative of the function Φ.Let us assume that third and higher order terms in this development are negligible with respect to the two first ones (which is the case for low σ^2 values) and let us note:

$$G = \left.\frac{\partial\Phi(\alpha)}{\partial\alpha}\right|_{\alpha=m_a} (a-m_a) \quad (4)$$

and

$$NG = \frac{1}{2}^T \left.\frac{\partial^2\Phi(\alpha)}{\partial\alpha^2}\right|_{\alpha=m_a} (a-m_a, a-m_a) \quad (5)$$

Under these conditions, the gaussian or non-gaussian nature of reflection coefficients is due to the second term NG. If the development were reduced to $k-m_k = G$, there would exist a linear relation between $k-m_k$ and $a-m_a$. These two vectors would both be gaussian. On the other hand, when the second order term is not negligible, the vector $k-m_k \simeq G+NG$ is no longer gaussian. Let us denote by M_X^k the X variable kth moment and let us consider the following distance between variables G and $G+NG$:

$$d(G, G+NG) = \sum_{k=1}^{+\infty} \frac{|M_G^k - M_{G+NG}^k|}{k!} = \sum_{k=1}^{+\infty} d_k \quad (6)$$

This distance comes from the l_1 norm applied to infinite sequences of the form $1, \frac{iM_X^1}{1!}, ..., \frac{(i)^n M_X^n}{n!}, ...$ which appears naturally in the development of the characteristic function in terms of its moments. It can only be used for variables satisfying "regularity" conditions that can be found in [5]. Under these conditions, it can be shown that

$$\lim_{\sigma^2 \to 0} d(G, G+NG) = 0$$

which can be used to show that reflection coefficient p.d.f. tends to the gaussian one when σ^2 tends to zero. For low σ^2 values, only the first terms in (6) can be considered. The convergence rate of reflection coefficient p.d.f. to the gaussian one depends on these terms, which are linked to AR model poles.

For example, let us consider the simple case of an order 2 AR model with two conjugated complex poles $p_1 = \rho e^{j\varphi}$ and $p_2 = \rho e^{-j\varphi}$. The second reflection coefficient, which is equal to the second AR parameter, is gaussian. Thus, only the first reflection coefficient has been considered. The first term in equation (6) corresponding to this coefficient can then be determined. The following results are obtained (see appendix):

$$d_1 = \sigma^2 \left| \frac{-2\rho c_{22} \cos\varphi}{(1+\rho^2)^3} - \frac{c_{12}}{(1+\rho^2)^2} \right| \qquad (7)$$

with $C_a = \sigma^2 \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}$. Higher order terms can be derived in a similar way. The $d(G, G+NG)$ variations as a function of ρ and φ can then be approximated by using the first terms in (6). For an order 2 AR model with $C_a = 10^{-2} \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$, the following results can be obtained :

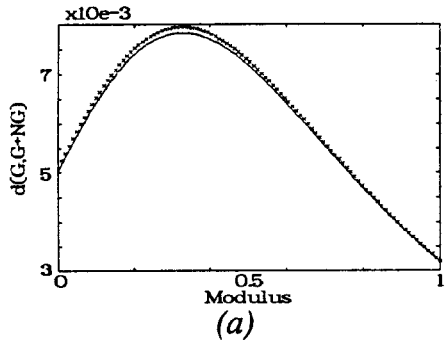

Fig. 1.(a) Variations of $d(G, G+NG)$ for the first reflection coefficient as a function of ρ $(\varphi = \frac{\pi}{4})$

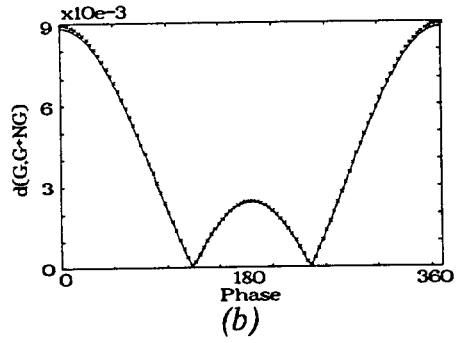

Fig. 1.(b) Variations of $d(G, G+NG)$ for the first reflection coefficient as a function of φ $(\rho = \frac{1}{2})$

In these figures, the solid and dashed curves represent respectively the variations of the first term given in (7) and the sum of the two first terms d_1+d_2. As can be seen in Fig. 1., there is little difference between the two curves. The higher the $d(G, G+NG)$, the lower the convergence of the reflection coefficient p.d.f. to the gaussian one.

To give an insight into how it compares with a conventional distance, let us plot the Kullback divergence between reflection coefficient p.d.f. (determined in the first part of the paper) and the gaussian one. The following results are then obtained :

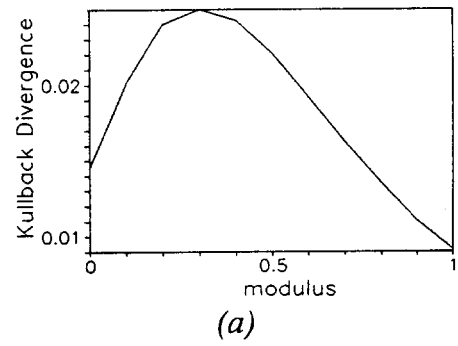

Fig. 2.(a) Kullback divergence between the first reflection coefficient p.d.f. and the gaussian one as a function of ρ $(\varphi = \frac{\pi}{4})$

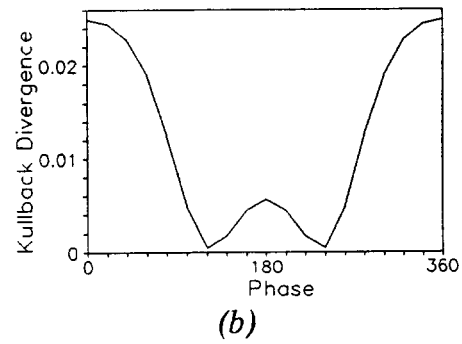

Fig. 2.(b) Kullback divergence between the first reflection coefficient p.d.f. and the gaussian one as a function of φ $(\rho = \frac{1}{2})$

It is easy to show that the qualitive behaviors of the two approaches are very similar. Numerous simulations allow us to think that the convergence of reflection coefficient distribution to the gaussian one (when parameter σ^2 tends to zero) is fast when the corresponding AR model poles are close to the unit circle.

4. CONCLUSION

The following new results are proposed in this paper:

- The first part is devoted to the determination of the reflection coefficient p.d.f. from that of AR parameters.
- This p.d.f. is then studied in the case of gaussian AR parameters, the covariance matrix of which is of the form $C_a = \sigma^2 M$, M being a constant matrix. It is well known that this p.d.f. converges towards the gaussian one when parameter σ^2 tends to zero. The convergence rate of this p.d.f. to the gaussian one is then studied as a function of the position of AR poles in the unit circle.

5. APPENDIX

The aim of this section is to determine the first moments of variables G and $G+N\dot{G}$ in the case of an order 2 AR model. Similar results can be obtained for higher orders. AR parameters and reflection coefficients are linked by the following relations:

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \Phi \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{a_1}{1+a_2} \\ a_2 \end{pmatrix}$$

which leads to:

$$m_k = \Phi(m_a) = \begin{pmatrix} \frac{m_1}{1+m_2} \\ m_2 \end{pmatrix}$$

with $m_a = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = E(a)$. Using equations (4) and (5), we then obtain :

$$G = \begin{pmatrix} \frac{a_1-m_1}{1+m_2} - \frac{m_1(a_2-m_2)}{(1+m_2)^2} \\ a_2 - m_2 \end{pmatrix}$$

$$NG = \begin{pmatrix} \frac{[m_1(a_2-m_2)^2-(a_1-m_1)(a_2-m_2)(1+m_2)]}{(1+m_2)^3} \\ 0 \end{pmatrix}$$

As can be seen, the second term in NG is equal to zero, which comes from the gaussianity of the second reflection coefficient. The moments of the first term in NG can be determined as a function of the AR parameter covariance matrix $C_a = \sigma^2 \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}$. For instance, using

$$E\left[(a_2 - m_2)^2\right] = c_{22}$$
$$E\left[(a_1 - m_1)(a_2 - m_2)\right] = c_{12}$$

allows us to determine the mean of NG:

$$E(NG) = \sigma^2 \begin{pmatrix} \frac{m_1 c_{22} - c_{12}(1+m_2)}{(1+m_2)^3} \\ 0 \end{pmatrix}$$

that is to say, as a function of the AR model poles:

$$E(NG) = \sigma^2 \begin{pmatrix} \frac{-2\rho c_{22} \cos\varphi}{(1+\rho^2)^3} - \frac{c_{12}}{(1+\rho^2)^2} \\ 0 \end{pmatrix}$$

It is well known that higher order moments of the Gaussian distribution can be determined as a function of its mean and covariance matrix . This allows us to compute, in a similar way, higher order moments of NG.

6. REFERENCES

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