Uniqueness Study of Measurements Obtainable with an Electromagnetic Vector Sensor

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Abstract: In this paper, we investigate linear dependence of steering vectors of one electromagnetic vector sensor. We show that every 3 steering vectors with distinct DOA's are linearly independent. We also show that 4 steering vectors with distinct DOA's are linearly independent if the ellipticity angles of the signals associated with any 2 of the 4 steering vectors are distinct. We then establish that 5 steering vectors are linearly independent if exactly 2 or 3 of them correspond to circularly polarized signals with the same spin direction. Finally, we demonstrate that given any 5 steering vectors, then for any DOA there exists a steering vector which is linearly dependent on the 5 steering vectors.

1. Introduction

Estimating the directions-of-arrival (DOA's) of narrow-band electromagnetic (EM) signals is a subject of both theoretical interest and practical importance. Many existing methods for estimating DOA's exploit the phase delays of signals received at an array of *scalar* sensors with respect to the signal at a reference sensor [1]. Around 1980, researchers began to propose methods that utilize measurements containing both the phase delays and polarization information. It has been demonstrated that with an incorporation of polarization information, the ability to resolve closely-spaced signals improves greatly.

Very recently, Nehorai and Paldi [2] proposed estimating DOA's with vector sensors, where each provides measurements of the *complete* electric and magnetic fields induced at the sensor. The measurements of the complete electric and magnetic fields that a vector sensor yields, offer much more than polarization information. Indeed, the cross product of the electric field and the magnetic field which an EM signal induces at a vector sensor gives directly the DOA of the EM signal. In comparison, one requires more than three appropriately deployed scalar sensors to determine uniquely both the azimuth and elevation of an EM signal.

Significant results on DOA estimation with vector-sensor arrays are available in the papers by Nehorai and Paldi [2], [3] and that by Li [4]. In [2], [3], the authors derived a data model which include the complete electromagnetic measurements. Through an explicit evaluation of the Cramér-Rao bound (CRB) on the estimation errors, they demonstrated the advantage of using vector sensors, and provided insight into the quality of DOA and polarization estimates. A cross product based DOA estimation algorithm for one signal was also proposed in [2], [3]. On estimating the DOA's of multiple signals with multiple vector sensors, Li [4] capitalized on the invariance properties among the sensor outputs to devise a powerful ESPRIT-based algorithm.

The studies in [2]–[4] were carried out based on the assumption that the steering vectors corresponding to the signals of concern are linearly independent. However, the validity of this assumption has yet to be established. As a matter of fact, linear independence of steering vectors relates very closely to uniqueness in DOA estimates. For the case of scalar-sensor arrays, Wax and Ziskind [5] and Nehorai *et al.* [6] have established a relationship among linear dependence of steering vectors, correlation among signals, and the number of signals whose DOA's can be uniquely determined.

In this paper, we shall present some results on linear dependence of steering vectors. These results confirm the validity of the assumption made in [2]–[4], and are useful for establishing the number of signals whose DOA's can be uniquely determined.

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2. Data Model and Preliminary Discussion

Since we are most concerned with DOA estimation with electromagnetic vector sensors, we shall refer to *electromagnetic vector sensor* as *sensor*, unless otherwise specified. We shall adopt the phasor data model developed in [2], [3]:

$$\mathbf{y}(t) = \sum_{k=1}^{n} (\mathbf{d}_k \otimes \mathbf{I}_6) \begin{pmatrix} \mathbf{I}_3 \\ (\mathbf{u}_k \times) \end{pmatrix} \mathbf{V}_k \mathbf{x}_k(t) + \mathbf{e}(t), \quad (1)$$

where $\mathbf{y}(t)$ and $\mathbf{n}(t)$ are $6m \times 1$ complex vectors respectively given by

$$\mathbf{y}(t) = [\mathbf{y}^{(1)}(t), \ldots, \mathbf{y}^{(m)}(t)]^T,$$

$$\mathbf{e}(t) = [\mathbf{e}^{(1)}(t), \ldots, \mathbf{e}^{(m)}(t)]^T,$$

$$\mathbf{d}_k = [e^{-j\omega_c \tau_{1,k}}, \ldots, e^{-j\omega_c \tau_{m,k}}]^T, \quad (2)$$

$$(\mathbf{u}_k \times) \triangleq \begin{pmatrix} 0 & -u_z^{(k)} & u_y^{(k)} \\ u_z^{(k)} & 0 & -u_x^{(k)} \\ -u_y^{(k)} & u_x^{(k)} & 0 \end{pmatrix},$$

$$\mathbf{u}_k = \begin{pmatrix} u_x^{(k)} \\ u_y^{(k)} \\ u_z^{(k)} \end{pmatrix} = \begin{pmatrix} \cos\phi_k \cos\psi_k \\ \sin\phi_k \cos\psi_k \\ \sin\psi_k \end{pmatrix},$$

$$\mathbf{V}_k = \begin{pmatrix} -\sin\phi_k & -\cos\phi_k \sin\psi_k \\ \cos\phi_k & -\sin\phi_k \sin\psi_k \\ 0 & \cos\psi_k \end{pmatrix},$$

$\mathbf{x}_k(t) \in C^{2\times 1}$, $\mathbf{I}_l$ is the $(l \times l)$ identity matrix, and $\otimes$ the Kronecker product. Note that $\mathbf{y}^{(k)}(t)$, $\mathbf{e}^{(k)}(t) \in C^{1\times 6}$, for $k = 1, \ldots, m$.

There is a physical meaning associated with each of the above variables. Indeed, a vector sensor will yield measurements of 3 perpendicular components of the electric and magnetic fields (in the same directions) that are induced at the sensor. We set up a Cartesian coordinate system, with the origin co-located with the reference sensor, and each of the axes in the direction of a component of the electric field measurable by the sensor. Then $\mathbf{y}^{(l)}(t)$ and $\mathbf{e}^{(l)}(t)$ are respectively the 6-component measurements of the electric and magnetic fields, and the noise at the lth sensor at time t. The symbol n denotes the number of EM signals impinging on the array, ω_c the frequency of the signals, ϕ_k and ψ_k are respectively the azimuth and elevation of the kth signal, and $\mathbf{u}_k$ is a unit vector pointing towards the DOA of the kth signal. (The values of ϕ_k and ψ_k are in $(-\pi, \pi]$ and $[-\pi/2, \pi/2]$ respectively.) The symbol m denotes the number of sensors, and $\tau_{l,k}$ is the differential delay of the kth signal at the lth sensor with respect to the reference sensor.

Two methods for transmitting signals, namely single signal transmission and dual signal transmission, are discussed in [2]. Here, we shall call signals transmitted using the former *polarized* signals, and the latter *general* signals (since with dual signal transmission, the signals can be unpolarized, partially polarized, or even polarized). For polarized signals, equation (1) can be written as

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{e}(t),$$

where

$$\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_n)],$$

$$\theta_k = (\phi_k, \psi_k, \alpha_k, \beta_k), \quad (3)$$

$$\mathbf{a}(\theta_k) = \mathbf{d}_k \otimes \mathbf{B}(\phi_k, \psi_k)\mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k), \quad (4)$$

$$\mathbf{B}(\phi_k, \psi_k) = \begin{pmatrix} -\sin\phi_k & -\cos\phi_k \sin\psi_k \\ \cos\phi_k & -\sin\phi_k \sin\psi_k \\ 0 & \cos\psi_k \\ -\cos\phi_k \sin\psi_k & \sin\phi_k \\ -\sin\phi_k \sin\psi_k & -\cos\phi_k \\ \cos\psi_k & 0 \end{pmatrix}, \quad (5)$$

$$\mathbf{Q}(\alpha_k) = \begin{pmatrix} \cos\alpha_k & \sin\alpha_k \\ -\sin\alpha_k & \cos\alpha_k \end{pmatrix}, \quad (6)$$

$$\mathbf{w}(\beta_k) = [\cos\beta_k, j\sin\beta_k]^T, \quad (7)$$

$$\mathbf{s}(t) = [s_1(t), \ldots, s_n(t)]^T,$$

and $s_l(t)$ is the complex envelope of the lth signal at time t. Note that $\alpha_k \in (-\pi/2, \pi/2]$ is referred to as orientation angle and $\beta_k \in [-\pi/4, \pi/4]$ ellipticity angle.

For general signals, equation (1) can be written as

$$\mathbf{y}(t) = \mathbf{A}_g \mathbf{s}_g(t) + \mathbf{e}(t),$$

where

$$\mathbf{A}_g = [\mathbf{d}_1 \otimes \mathbf{B}(\phi_1, \psi_1), \ldots, \mathbf{d}_n \otimes \mathbf{B}(\phi_n, \psi_n)],$$

$$\mathbf{s}_g(t) = [\tilde{s}_1^{(1)}(t), \tilde{s}_2^{(1)}(t), \ldots, \tilde{s}_1^{(n)}(t), \tilde{s}_2^{(n)}(t)]^T,$$

and $\tilde{s}_1^{(k)}(t)$ and $\tilde{s}_2^{(k)}(t)$ represent the complex envelopes of the kth transmitted signal.

It is easy to see that for a polarized signal with azimuth ϕ_k, elevation ψ_k, orientation angle α_k and ellipticity angle β_k, the array measurements (in the absence of noise) lie in the subspace spanned by the vector $\mathbf{a}(\theta_k)$ as defined in (4). For a scalar-sensor array, it is a well known fact that in the presence of only a signal, the array measurements will be confined to the subspace spanned by the *steering vector* corresponding to the DOA of the signal. Thus it makes good sense to refer to $\mathbf{a}(\boldsymbol{\theta}_k)$ as a steering vector. In comparison, the array measurements of a general signal will lie in a 2 dimensional subspace spanned by $\mathbf{d}_k \otimes \mathbf{B}(\phi_k, \psi_k)$, where (ϕ_k, ψ_k) is the DOA of the signal. Linear dependence of $\mathbf{a}(\boldsymbol{\theta}_k)$'s will be the main concern of this paper.

3. Linear Dependence of Steering Vectors

Observe that the steering vectors of a vector-sensor array (for polarized signals) are much more complex than those of a scalar-sensor array. Basically, for a polarized signal with azimuth ϕ_k, elevation ψ_k, orientation angle α_k and ellipticity angle β_k, the steering vector of a vector-sensor array corresponding to the signal has an additional term $\mathbf{B}(\phi_k, \psi_k)\mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k)$, compared with that of a scalar-sensor array having the same sensor configuration. The term $\mathbf{B}(\phi_k, \psi_k)\mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k)$ is a complicated vector given by

$$\begin{pmatrix} -(c_\alpha c_\beta + j s_\alpha s_\beta) s_\phi + (s_\alpha c_\beta - j c_\alpha s_\beta) c_\phi s_\psi \\ (c_\alpha c_\beta + j s_\alpha s_\beta) c_\phi + (s_\alpha c_\beta - j c_\alpha s_\beta) s_\phi s_\psi \\ (-s_\alpha c_\beta + j c_\alpha s_\beta) c_\psi \\ -(s_\alpha c_\beta - j c_\alpha s_\beta) s_\phi - (c_\alpha c_\beta + j s_\alpha s_\beta) c_\phi s_\psi \\ (s_\alpha c_\beta - j c_\alpha s_\beta) c_\phi - (c_\alpha c_\beta + j s_\alpha s_\beta) s_\phi s_\psi \\ (c_\alpha c_\beta + j s_\alpha s_\beta) c_\psi \end{pmatrix},$$

where the symbols s_α and c_α, denote respectively $\sin\alpha$ and $\cos\alpha$.

Basically, to examine whether a set of k steering vectors of a vector-sensor array are linearly independent, one has to analyze the rank of

$$\Big(\mathbf{d}_1 \otimes \mathbf{B}_1\mathbf{Q}_1\mathbf{w}_1, \ldots, \mathbf{d}_k \otimes \mathbf{B}_k\mathbf{Q}_k\mathbf{w}_k\Big),$$

where $\mathbf{d}_l$, $\mathbf{B}_l = \mathbf{B}(\phi_l, \psi_l)$, $\mathbf{Q}_l = \mathbf{Q}(\alpha_l)$, and $\mathbf{w}_l = \mathbf{w}(\beta_l)$ are as defined in (2), (5), (6) and (7) respectively. For the same purpose with scalar-sensor arrays, one just has to examine the rank of $[\mathbf{d}_1, \ldots, \mathbf{d}_k]$, which is apparently simpler. Fortunately, as will be seen in Theorem 1, it suffices to analyze the rank of

$$\Big(\mathbf{d}_1 \otimes \mathbf{B}(\phi_1, \psi_1), \ldots, \mathbf{d}_k \otimes \mathbf{B}(\phi_k, \psi_k)\Big),$$

which contains fewer terms due to the absence of $\mathbf{Q}(\alpha)\mathbf{w}(\beta)$'s. As a result, the complexity of the analysis can be reduced significantly.

Theorem 1: Every k steering vectors of a vector-sensor array with distinct DOA's, are linearly independent if and only if for every set of k steering vectors with distinct DOA's $(\phi_1, \psi_1), \ldots, (\phi_k, \psi_k)$,

$$rank\ \Big(\mathbf{d}_1 \otimes \mathbf{B}(\phi_1, \psi_1), \ldots, \mathbf{d}_k \otimes \mathbf{B}(\phi_k, \psi_k)\Big) = 2k.$$

(The proof of this and other theorems appear in [7].)

From Theorem 1, we obtain the following corollary.

Corollary to Theorem 1: For an m-sensor array, there exist $(3m + 1)$ steering vectors with distinct DOA's that are linearly dependent.

Although this corollary is an immediate consequence of Theorem 1, it offers more explicitly, the maximum number of steering vectors that can be linearly independent for a general array. Indeed, since each steering vector has $6m$ elements, it is possible that every $6m$ different steering vectors are linearly independent. This corollary, however, states that there exist $(3m+1)$ (which is less than $6m$) steering vectors, corresponding to distinct DOA's, which are linearly dependent.

Applying Theorem 1, we obtain the following theorem on linear independence of 3 steering vectors.

Theorem 2: For any vector-sensor array, every 3 steering vectors with distinct DOA's are linearly independent.

Theorem 2, together with the fact that there exist 4 steering vectors of one vector sensor that are linearly dependent (see the Corollary to Theorem 1), are useful for establishing the maximum number of signals whose DOA's can be uniquely determined.

The next theorem states that the condition for 4 steering vectors to be linearly dependent is very stringent.

Theorem 3: Four steering vectors of a vector-sensor array corresponding to distinct DOA's, are linearly *dependent* only if the ellipticity angles of the signals corresponding to the steering vectors are identical.

We also establish a condition for 5 steering vectors to be linearly independent.

Theorem 4: Five steering vectors of a vector-sensor array with distinct DOA's are linearly independent if exactly k of them correspond to circularly polarized signals with the same spin, where $k \in \{2, 3\}$.

The conditions for 4 or 5 steering vectors to be linearly independent can be useful when considering specific applications. For example, on estimating the DOA's of skywaves [8], one can assume that practically every 4 steering vectors are linearly independent. Indeed, when polarized signals from a transmitter are reflected from the various layers of the ionosphere, the polarizations of the reflected signals tend to vary with, among other factors, the electron densities of the ionosphere around the layers. Since the electron densities of different layers of the ionosphere are likely to be different, one can expect the polarizations of reflected signals from the various layers to be distinct. Thus, it follows from Theorem 3 that every 4 steering vectors corresponding to the signals are linearly independent.

Next, we establish a theorem that provides a good insight into estimating the DOA's of 5 uncorrelated signals.

Theorem 5: Given *any* 5 steering vectors of one vector sensor, then for *any* DOA there exists a steering vector which is linearly dependent on the 5 steering vectors.

It is immediate from Theorem 5 that in the presence of 5 uncorrelated signals, one can find a steering vector corresponding to an arbitrary direction, that intersects the signal subspace. This means that estimation of DOA's of 5 signals with one EM vector sensor is impossible.

Finally, when a set of steering vectors are circularly polarized, there is an interesting characterization of their linear dependence.

Theorem 6: Every 4 steering vectors of one vector sensor corresponding to circularly polarized signals having the same spin direction are linearly dependent.

4. Conclusion

We showed that the task of establishing the existence of linearly dependent steering vectors may be simplified greatly via a decoupling of the DOA parameters from the polarization parameters. We then established that every 3 steering vectors with distinct DOA's are linearly independent. We also showed that 4 steering vectors with distinct DOA's are linearly independent if the ellipticity angles of the signals associated with any 2 of the 4 steering vectors are distinct. We next established that 5 steering vectors are linearly independent if exactly 2 or 3 of them correspond to circularly polarized signals with the same spin direction. Finally, we demonstrated that given any 5 steering vectors, then for any DOA there exists a steering vector which is linearly dependent on the 5 steering vectors.

So far, we have focused our study on a single vector sensor. There are many interesting and important issues regarding linear dependence of steering vectors of a multiple-sensor array. First, given a multiple-sensor array, it is of practical importance to identify the maximum number ν associated with the array, where every ν steering vectors with distinct DOA's are linearly independent. Second, it is crucial to establish the conditions under which a set of steering vectors are linearly independent. Third, it is a challenging task to identify sensor configurations giving rise to steering vectors with high order of linear independence. In a companion paper [9], we address some of these issues.

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