ARRAY PROCESSING USING PARAMETRIC SIGNAL MODELS

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ABSTRACT

This paper attempts to assess the potential performance gain of spatial-temporal processing relative to conventional spatial processing, for signals obeying a deterministic parametric model. The Cramer-Rao bound (CRB) on the estimates of the source directions of arrival (DOA) is used to quantify this gain. Spatial-temporal processing does not yield any such gain in the single source case, or for multiple *coherent* signals. However, significant gains can be achieved for multiple non-coherent signals.

1. INTRODUCTION

Conventional array processing methods are based on the spatial properties of the signals impinging on the array and ignore their temporal structure. In many applications including communications, sonar, radar and Doppler ultrasound, the signals have a known temporal structure which can be used to enhance the performance of array processing methods. Based on this observation, several authors have recently advocated the use of signal-selective array processing methods which exploit both temporal and spatial properties of the signals. For example, [1] presents a technique based on high-order statistics utilizing the non-Gaussian nature of the signals. Techniques exploiting the cyclostationarity of digital communications signals are presented in [2].

In this paper we take a different approach and consider signals whose temporal variation can be described by a known function of time with unknown deterministic parameters. In other words we assume that each of the source signals obeys a deterministic parametric model. This class of signals is very general and includes as a special case the class of polynomial phase signals [3] which is quite general by itself. Simple examples of polynomial phase signals are the linear and quadratic frequency modulated (FM) signals commonly used in radar and sonar. Another example includes signals which have a continuous and smooth phase and a slowly varying amplitude. Sufficiently short segments of such signals can be well approximated by low order polynomials. This type of signals is often used in analog communication systems.

The purpose of this paper is to assess the potential gain in spatial-temporal processing relative to conventional spatial processing, for the above class of deterministic signals. It appears likely that prior information about the temporal structure of the signals will yield some gain in performance. By deriving the CRB on the estimates of the source directions of arrival we quantify this gain and identify the cases for which the gain is significant.

2. PROBLEM FORMULATION

We consider an arbitrary array composed of M sensors. Let N plane waves impinge on the array from directions $\{\theta_1, \ldots, \theta_N\}$. Each of the source signals is completely characterized by P parameters and is given by $s(t, \mathbf{b}_n)$ where $\mathbf{b}_n$ is the vector of the n-th signal parameters and $s()$ is a known complex function.

The noise free signal at the output of the m-th element is given by

$$y_m(t) = \sum_{n=1}^{N} s[t + \tau_m(\theta_n), \mathbf{b}_n] \qquad (1)$$

where $\tau_m(\theta)$ is the differential propagation delay from a source at direction θ to the m-th element.

$s_n(t) = s(t, \mathbf{b}_n)$ can be written as $u_n(t) \exp^{j\phi_n(t)}$ where $u_n(t)$ and $\phi_n(t)$ are the amplitude and phase functions. Usually $\phi_n(t)$ and $u_n(t)$ are continuous and smooth function of time and $u_n(t)$ varies slowly compared to $\phi_n(t)$.

Assuming that the variation of the instantaneous frequency, $\omega_n(t) = \frac{\partial \phi_n(t)}{\partial t}$, is a small fraction of the

This work was supported by the National Science Foundation under grant no. III-9360166.

0-7803-2431-5/95 $4.00 © 1995 IEEE

carrier frequency ω_0, and that $u_n(t)$ varies slowly compared to $\phi_n(t)$, we can make the following approximation.

$$s[(t+\tau_m(\theta_n)),\mathbf{b}_n] = s(t,\mathbf{b}_n)\exp(j\omega_0\tau_m(\theta_n)) \quad (2)$$

Without loss of generality we assume that the array outputs are down converted to baseband prior to sampling. The vector of received signals can be written as

$$\mathbf{x}(t_k) = A(\boldsymbol{\theta})\mathbf{q}(t_k,\mathbf{b}) + \mathbf{n}(t_k) \qquad 1 \le k \le K. \quad (3)$$

where $\mathbf{n}(t)$ is the vector of measurement noise,

$$\mathbf{q}(t,\mathbf{b}) = [q(t,\mathbf{b}_1),\ldots,q(t,\mathbf{b}_n)]^T$$
$$q(t,\mathbf{b}_n) = s(t,\mathbf{b}_n)\exp\{-j\omega_0 t\} \quad (4)$$

and

$$A(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1),\ldots,\mathbf{a}(\theta_N)] \quad (5)$$

$\mathbf{a}(\theta)$ is the array manifold in the direction θ at frequency ω_0,

$$\mathbf{a}(\theta) = [\exp(j\omega_0\tau_1(\theta)),\ldots,\exp(j\omega_0\tau_M(\theta))]^T \quad (6)$$

We assume that the noise vector $\mathbf{n}(t)$ is Gaussian distributed and satisfies $E[\mathbf{n}(t)] = 0$, $E[\mathbf{n}(t)\mathbf{n}^T(s)] = 0$ and

$$E[\mathbf{n}(t)\mathbf{n}^H(s)] = \begin{cases} \eta I & t = s \\ 0 & t \ne s \end{cases} \quad (7)$$

3. THE CRB

Denote the vector of unknown parameters by $\boldsymbol{\psi}$,

$$\boldsymbol{\psi} = [\eta, b_{11},\cdots,b_{N1},\cdots,b_{1P},\cdots,b_{NP},\boldsymbol{\theta}^T]^T \quad (8)$$

where b_{np} is the p-th temporal parameter of the n-th source.

It is well known that the CRB for $\boldsymbol{\psi}$ is given by

$$\mathrm{CRB}(\boldsymbol{\psi}) = [F(\boldsymbol{\psi})]^{-1} \quad (9)$$

where $F(\boldsymbol{\psi})$ is the Fisher information matrix (FIM) for $\boldsymbol{\psi}$. We have shown that

$$F(\boldsymbol{\psi}) = \frac{2}{\eta}\mathrm{Re} \quad (10)$$
$$\begin{bmatrix} \frac{MK}{2\eta} & 0 & \cdots & 0 & 0 \\ 0 & A^HA \times R_{11}^* & \cdots & A^HA \times R_{1P}^* & A^HD \times R_1^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & A^HA \times R_{P1}^* & \cdots & A^HA \times R_{PP}^* & A^HD \times R_P^T \\ 0 & D^HA \times R_1^* & \cdots & D^HA \times R_P^* & D^HD \times R^* \end{bmatrix}$$

where $\times$ denotes the Hadamard product and the n-th column of D is $\mathbf{d}(\theta_n) = \frac{\partial \mathbf{a}(\theta)}{\partial \theta}|_{\theta=\theta_n}$.

The matrices R, R_p and R_{ps} for $p,s = 1,\ldots,P$ are N-by-N with elements,

$$(R)_{nl} = \sum_{k=1}^{K} q_n(t_k)q_l^*(t_k)$$
$$(R_p)_{nl} = \sum_{k=1}^{K} q_n(t_k)\frac{\partial q_l^*(t_k)}{\partial b_{lp}}$$
$$(R_{ps})_{nl} = \sum_{k=1}^{K} \frac{\partial q_n(t_k)}{\partial b_{np}}\frac{\partial q_l^*(t_k)}{\partial b_{ls}} \quad (11)$$

These matrices summarize the temporal characteristics of the sources. In the following we shall refer to them as the source matrices.

To assess the potential gain that can be achieved by spatial-temporal processing relative to conventional spatial processing we compare the bound given by (9)–(10) to the CRB for the case where no prior information is available on the temporal structure of the signal. In this case the samples of the source signals are assumed to be unknown deterministic parameters varying from snapshot to snapshot. This bound was derived in [4] and is given by,

$$\mathrm{CRB}_c^{-1}(\boldsymbol{\theta}) = \frac{2}{\eta}\mathrm{Re}\{D^H[I - A(A^HA)^{-1}A^H]D \times R^T\} \quad (12)$$

We will refer to this bound as the *conventional bound* and to the bound given by (9)–(10) as the *model-based* bound.

The single source conventional bound is given by,

$$\text{CRB}_c^{-1}(\theta) = \frac{2}{\eta E} \frac{1}{\mathbf{d}^H(\theta)\mathbf{d}(\theta) - \frac{1}{M}|\mathbf{d}^H(\theta)\mathbf{a}(\theta)|^2} \quad (13)$$

where $E \sum_{k=1}^{K} |q(t_k)|^2$ is the signal energy.

First, consider the case where the source matrices are diagonal. In this case each source signal and its derivatives with respect to the model parameters are orthogonal to all other source signals and their derivatives. This orthogonality assumption may appear quite restrictive. However, it can be shown that for polynomial phase signals the source matrices are at least approximately diagonal as long as there is no pair of identical (or almost identical) signals. Since Short segments of signals which have a continuous and smooth phase and a slowly varying amplitude can be approximated by polynomial phase signals, this assumption holds for a large class of signals.

By reordering the parameters in $\boldsymbol{\psi}$ so that all parameters corresponding to a certain source are grouped together the FIM becomes block diagonal. In this case, therefore, the model based CRB for the parameters of the n-th source is simply the single source model based bound. This is not surprising. When the orthogonality condition above holds, the sources can be resolved by temporal processing yielding the single source bound. In contrast with the model based bound, the conventional bound is not reduced to the single source bound in this case.

On the other hand we have shown that the single source model based bound for the source direction is the same as the single source conventional bound. Thus, in the single source case spatial-temporal processing does not yield any gain in performance of the direction estimation relative to conventional spatial methods.

Next, consider the other extreme where all source signals are the same up to a complex amplitude. We shall follow a common practice and refer to such signals as coherent signals. In this case we have shown that $\text{CRB}(\boldsymbol{\theta}) \geq \text{CRB}_c(\boldsymbol{\theta})$ where $\text{CRB}_c(\boldsymbol{\theta})$ is the conventional bound. It follows that in the case of coherent sources, spatial-temporal processing cannot yield any gain in performance relative to conventional spatial methods. Again, this result is not surprising. When the source signals are coherent they cannot be resolved by temporal processing. Then the bound coincides with the conventional bound which predicts the performance of spatial methods.

We now consider the intermediate case where the source matrices have full rank, but are not diagonal. For this case we will not attempt to obtain analytical results for arbitrary source geometry. Instead, we will examine the case where all sources have the same direction. In this case the signals cannot be resolved by spatial processing.

Theorem 1 Denote by $\mathbf{R}$ the following matrix,

$$\mathbf{R} = \begin{bmatrix} R_{11} & \cdots, & R_{1P} \\ \vdots & \vdots & \vdots \\ R_{P1} & \cdots, & R_{PP} \end{bmatrix} \quad (14)$$

Assume that $\mathbf{R}$ is non-singular. Let N sources impinge on the array from directions $\{\theta_1, \ldots, \theta_N\}$. Denote by $\text{CRB}(\theta_n)$ the model based bound for the direction of the n-th source. If $\mathbf{a}(\theta_n) = \mathbf{a}(\theta)$ for $n = 1, \cdots, N$

$$\text{CRB}(\theta_n) = \frac{2}{\eta E} \frac{1}{\mathbf{d}^H(\theta_n)\mathbf{d}(\theta_n) - \frac{1}{M}|\mathbf{d}^H(\theta_n)\mathbf{a}(\theta_n)|^2} \quad (15)$$

which is the single source bound. The proof is given in [5].

The above theorem suggests that as long as the source matrices are not singular or close to singular, spatial-temporal processing can approach the single source performance. This conjecture is verified by numerical experiments in the following section.

4. POLYNOMIAL PHASE SIGNALS

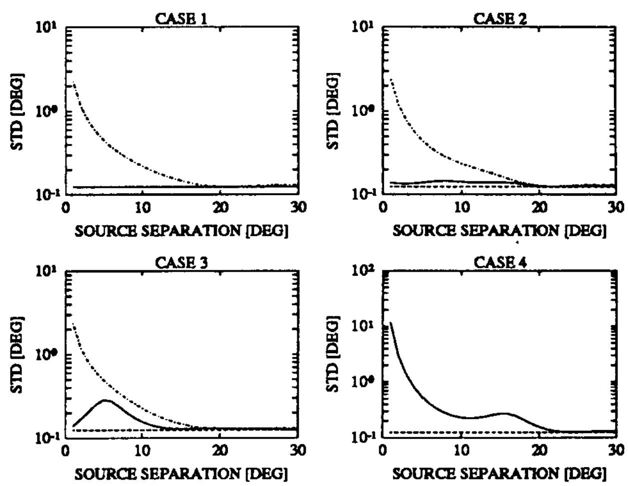

Figure 1: Bounds for the DOA standard deviation for quadratic FM signals. Model based two source CRB (solid), Single source CRB (dashed), Conventional CRB (dash-dot).

In this section we assume that $u_n(t)$ is constant in time and $\phi_n(t)$ can be approximated as a polynomial function of time. Then the baseband n-th signal is given by

$$q_n(t) = \alpha_n \exp\{j \sum_{p=0}^{P-2} \gamma_{np} t^p\} \quad (16)$$

We consider three bounds: The exact model based bound given by (10), the approximate model based bound obtained by replacing the source matrices by their diagonal estimates and the conventional bound. Note that the approximate model based bound for each of the source directions is simply the single source model based bound for that source.

We study the CRB for the following pair of quadratic FM signals.

$$q_n(t) = \alpha \exp\{j2\pi\xi_n f_s(-t + \frac{4}{T}t^2 - \frac{8}{3T^2}t^3)\} \quad (17)$$

where $n = 1, 2$, f_s is the sampling frequency and $\xi_1 = 0.4$. ξ_2 assumes the values $\{-0.4, 0.39, 0.396, 0.4\}$ in cases 1,2,3,4 respectively. We assume that the antenna array is an equispaced linear array. The array is composed of 8 elements with inter-element spacing of $\lambda/2$ where λ is the wavelength of the carrier. Note that the array beamwidth is approximately 13°. The number of sources is known and equal to 2, and the number of available snapshots is 512. The direction of the first source is fixed at 0° while the direction of the second source is varied according to the source separation. Results are given for the first source.

The results are shown in Figure 1 where we fix the SNR at 0 dB and plot the standard deviation as a function of the source separation. Case 1 represents the case where the source matrices are approximately diagonal. Note that in this case the signals occupy the same bandwidth $[-0.4f_s, 0.4f_s]$. We observe that the approximate model-based bound coincides with the exact bound, verifying that, in this case, the exact bound can be approximated by the single source bound. The standard deviation predicted by the model-based bounds is significantly lower than the standard deviation predicted by the conventional bound. The gain in performance increases for decreasing source separation. In this case, As predicted by the results of Section 3, spatial-temporal methods can yield significant gain in performance relative to spatial based methods. In case 2 the source matrices are not diagonal but are not close to singular matrices. In this case the model based bound predicts near single source performance as anticipated by our earlier conjecture. In case 4 the source signals are coherent and the model based bound coincides with the conventional bound. Not surprisingly, intermediate results are obtained for the intermediate case.

5. CONCLUSIONS

Assuming that the source signals obey a deterministic parametric model, we used the Cramer Rao bound to study the potential gain in spatial-temporal processing vs. conventional spatial processing. We have shown that for the single-source case, spatial-temporal processing cannot yield any gain in performance relative to conventional spatial methods. For multiple non-coherent signals, incorporating temporal processing can achieve the single-source performance, yielding a significant gain for the case of multiple sources with small spatial separation relative to the beamwidth of the array. However, spatial-temporal processing cannot yield any gain in performance for multiple coherent signals.

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