# The Best Achievable Performance for DOA Estimation Using ESPRIT Arrays

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## Abstract

This paper studies the Cramér-Rao (CR) bound for the problem of estimating the directions of arrival of narrow-band plane waves impinging on an ESPRIT array with multiple displacement invariances. We call this bound the MI-CR bound. The CR bound for ESPRIT arrays with a single invariance was derived in [1], and called the ESPRIT-CR bound. The difference between the ESPRIT-CR bound and the MI-CR bound can be large, especially for highly correlated sources. We give examples to show that a recently proposed algorithm called WSE achieves the MI-CR bound derived in this paper.

## 1 Introduction

ESPRIT [2] is a popular subspace-based approach to array signal processing and other related parameter estimation problems. Since it was proposed, ESPRIT has received much attention because of its robustness and computational efficiency. The original ESPRIT is formulated assuming that an array consists of two identical subarrays, separated by a known displacement vector. This structure arises naturally in applications such as radar, radio communication, underwater acoustics, seismology and etc., where uniform linear arrays (ULAs) are commonly employed. The original formulation of ESPRIT only takes advantage of a single invariance of the array. When multiple invariances exist, such as in a ULA, the ESPRIT algorithm can be applied by using overlapping subarrays. But, the additional structure of the array can not be optimally exploited. Thus, before any effort is made to improve ESPRIT, it is of great interest to study the best achievable performance, i.e., to derive the Cramér-Rao bound based on the multiple invariance ESPRIT parametrization. In [1], an ESPRIT-CR bound is derived under the assumption that only a single invariance exists. In this case, it is shown by numerical examples that the ESPRIT algorithm is asymptotically efficient. In [3], an algorithm for estimating the directions of arrival of narrow band signals with multiple invariance ESPRIT arrays was proposed. But [3] does not contain any CR bound other than the one for the case of a uniform linear array with identical sensors. In this paper, we provide more general CR bounds for multiple invariance ESPRIT arrays. Numerical example shows that, for this case, the ESPRIT algorithm is not efficient, but the derived MI-CR bound is attained by a recently proposed algorithm called Weighted Subspace Estimation (WSE) [4, 5].

## 2 Basic Assumptions

Assume that there are $p$ narrow-band plane waves impinging on an $L$-sensor array. The narrow-band assumption makes it possible to represent the propagation delays as simple phase shifts. The output of the array is the superposition of the individual emitter signals, weighted by the array response, and can be expressed by the following model:

$$\tilde{\mathbf{y}}(k) = \mathbf{A}(\theta)\mathbf{s}(k) + \mathbf{n}(k), \qquad k = 1, \cdots, K, \tag{1}$$

where $\tilde{\mathbf{y}}(k) \in \mathbf{C}^{L\times 1}$ is the noisy observation vector, $\mathbf{n}(k)$ is noise vector sampled from a random process which is uncorrelated with signals, the matrix $\mathbf{A} \in \mathbf{C}^{L\times p}$ contains the array response (steering) vectors, and $\mathbf{s}(k) \in \mathbf{C}^{p\times 1}$ is the unknown vector of wave amplitudes. It is assumed that only the azimuth angle is of interest, leading to a one parameter (per source) problem. Thus, the unknown DOA vector to be estimated from the array output is

$$\theta = [\theta_1 \;\; \theta_2 \;\; \cdots \;\; \theta_p]. \tag{2}$$

In an ESPRIT array with only one invariance, the array is made up of two identical subarrays. They are displaced by a known translation vector $\Delta$. The information about DOAs is embedded in the following matrix of phase delays between two subarrays for the $p$ wavefronts,

$$\Phi = \begin{pmatrix} r_1 e^{j(2\pi/\lambda)\Delta \sin\theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_p e^{j(2\pi/\lambda)\Delta\sin\theta_p} \end{pmatrix}. \tag{3}$$

In the above equation, $r_i = 1$ for $i = 1, \cdots, p$, $\Delta = \|\Delta\|$ and $\lambda$ is the wavelength of the incoming signals.

In an array with multiple invariances, a subset $S^1$ contains sensors with unknown response patterns. The array contains other subsets of sensors, $S_1^1, S_2^1, \cdots, S_{q1}^1$ with



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the following two properties. First, the sensors in each $S_i^1$ are physically displaced from the sensors in $S^1$ in the direction of a fixed translation vector $\Delta_1$ ( the length of $\Delta_1$ corresponds to the smallest translation). Second, the sensors in each $S_i^1$ have the same responses as the corresponding sensors in $S^1$. The number of sensors in each $S_i^1$ must be less than or equal to the number fo sensors in $S^1$. Also, the array may have other sets of sensors, $\{S^m, S_1^m, S_2^m, \cdots, S_{q_m}^m\}$, $m = 1, \cdots, M$, satisfying the two properties described above.

The emitter signals in (1), $\mathbf{s}(k)$, are modeled as zero-mean, complex Gaussian, temporally white random variables with covariance matrix

$$\mathbf{\Omega}_s = E[\mathbf{s}(k)\mathbf{s}^H(k)]. \tag{4}$$

It is assumed that $\mathbf{\Omega}_s$ is a completely unknown Hermitian matrix. The noise sequence $\mathbf{n}(k)$ also has zero mean and is both temporally and spatially white with variance $\sigma^2$. The array output covariance matrix is then given by

$$\mathbf{R} = E[\tilde{\mathbf{y}}(k)\tilde{\mathbf{y}}^H(k)] = \mathbf{A}\mathbf{\Omega}_s\mathbf{A}^H + \sigma^2\mathbf{I}. \tag{5}$$

We assume that both the signal covariance matrix, $\mathbf{\Omega}_s$, and the steering matrix, $\mathbf{A}$, are of full rank; thus, coherent (completely correlated) signals are not allowed. Also, the steering matrix $\mathbf{A}$ is uniquely determined, up to an arbitrary scaling of the columns [1]. For convenience, the elements of the first row of $\mathbf{A}$ are customarily fixed to unity. In addition, the requirement for no ambiguities when obtaining the DOA estimates from the eigenvalues of $\mathbf{\Phi}$ determines the range of DOA's. From (3), we see that to uniquely obtain the DOA estimates, the arguments of the eigenvalues of $\mathbf{\Phi}$ must satisfy

$$-\pi \le \frac{2\pi}{\lambda}\Delta \sin\theta_i \le \pi. \tag{6}$$

For a multiple-invariance array, $\Delta$ in (6) is the norm of the smallest translation vector in the array. If the range of DOA's is $(-\pi/2, \pi/2)$, this inequality implies

$$\Delta \le \frac{\lambda}{2}. \tag{7}$$

Based on the assumptions stated above, we now proceed to derive the CR bound for a multiple invariance array.

## 3 MI-CR Bound

Under the Gaussian signal waveform assumption, the array output is a stationary, temporally white, zero-mean complex Gaussian random process with covariance matrix $\mathbf{R}$ given in (5). The normalized negative log likelihood function of the observations $\tilde{\mathbf{y}}(1), \cdots, \tilde{\mathbf{y}}(K)$, has the following form:

$$l(\xi) = L\log\pi + \log|\mathbf{R}(\xi)| + \mathrm{tr}\{\mathbf{R}^{-1}(\xi)\hat{\mathbf{R}}\}, \tag{8}$$

where $\xi$ is the unknown parameter vector, and $\hat{\mathbf{R}}$ is the sample covariance matrix of the noisy array outputs defined as

$$\hat{\mathbf{R}} = \frac{1}{K}\sum_{k=1}^{K}\tilde{\mathbf{y}}(k)\tilde{\mathbf{y}}^H(k). \tag{9}$$

It can then be shown that the Cramér-Rao inequality is implicitly given by

$$K \times E[(\hat{\xi} - \xi_0)(\hat{\xi} - \xi_0)^T] \ge \mathbf{CRBM} \tag{10}$$

$$(\mathbf{CRBM}^{-1})_{ij} = \mathrm{tr}\{\mathbf{R}^{-1}\mathbf{R}_{\xi_i}\mathbf{R}^{-1}\mathbf{R}_{\xi_j}\}, \tag{11}$$

where $(\mathbf{CRBM}^{-1})_{ij}$ is the $ij$the element of $\mathbf{CRBM}^{-1}$.

Thus, the key for the calculation of the MI-CR bound is to obtain a correct parametrization of the steering matrix $\mathbf{A}$ (or $\mathbf{R}$), determine the unknown parameter vector, and then compute the first partial derivative of $\mathbf{R}$ with respect to the unknown parameters. We provide the following procedure to do this:

1. Find the phase delay matrix $\mathbf{\Phi}_i$ for each of the sensor sets $S^i$ described in the previous section,

$$\mathbf{\Phi}_i = \begin{pmatrix} r_1 e^{j(2\pi/\lambda)\Delta_i \sin\theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_p e^{j(2\pi/\lambda)\Delta_i \sin\theta_p} \end{pmatrix}. \tag{12}$$

The true values of $r_i$, $i = 1, \cdots, p$ are all equal to 1.

2. Let $\mathcal{A}_i$ be the array response matrix for the sensors in $S^i$. Let $\Delta_{ij} = \delta_{ij} \cdot \Delta_i$ be the displacement vectors for the sensors in $S_j^i$; by convention, $\delta_{i1} = 1$. Then, the general form of the steering matrix $\mathbf{A}$ can then be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_M \end{bmatrix}, \tag{13}$$

$$\mathbf{A}_i = \begin{bmatrix} \mathcal{A}_i \\ \mathbf{J}_{i1}\mathcal{A}_i\mathbf{\Phi} \\ \mathbf{J}_{i2}\mathcal{A}_i\mathbf{\Phi}^{\delta_{i2}} \\ \vdots \\ \mathbf{J}_{iq_i}\mathcal{A}_i\mathbf{\Phi}^{\delta_{iq_1}} \end{bmatrix}, \tag{14}$$

where each $\mathbf{J}_{ij}$ is either an identity matrix , or some subset of the rows of an identity matrix which is used for the case where some $S_j^i$ has fewer sensors than $S^i$.

3. Determine the unknown parameter vector $\xi$.

Using $\bar{x}$ to denote the real part of the element $x$ and $\tilde{x}$ for the imaginary part of $x$, the unknown parameters of the steering matrix $\mathbf{A}$ can be collected in the following vector,

$$\xi_A = \begin{bmatrix} \text{vec}(\bar{\mathcal{A}}_1^!) \\ \text{vec}(\check{\mathcal{A}}_1^!) \\ \text{vec}(\bar{\mathcal{A}}_2) \\ \text{vec}(\check{\mathcal{A}}_2) \\ \vdots \\ \text{vec}(\bar{\mathcal{A}}_M) \\ \text{vec}(\check{\mathcal{A}}_M) \end{bmatrix}, \quad (15)$$

where $\mathcal{A}_1^!$ is $\mathcal{A}_1$ excluding the first row. This is because the first row of the steering matrix is fixed to unity as discussed in the previous section.

The whole unknown parameter set then includes $\xi_A$, the parameters from $\mathbf{\Phi}_i$, the components of $\mathbf{\Omega}_s$, which are

$$\begin{aligned} s_i &= \{\Omega s\}_{ii},\ i = 1, \cdots, p & (16) \\ \bar{s}_{ij} &= \{\bar{\Omega}_s\}_{ij},\ i = 1, \cdots, p,\ j = i+1, \cdots, p & (17) \\ \check{s}_{ij} &= \{\check{\Omega}_s\}_{ij},\ i = 1, \cdots, p,\ j = i+1, \cdots, p & (18) \end{aligned}$$

and $\sigma^2$.

In the case of $M = 1$, the unknown parameter vector can be expressed as

$$\begin{aligned} \xi = [\ & \sigma^2,\ r_i, \theta_i,\ i = 1, \cdots, p, \\ & \bar{a}_{ij}, \check{a}_{ij},\ i = 2, \cdots, n,\ j = 1, \cdots, p, \\ & s_i, \bar{s}_{ij}, \check{s}_{ij},\ i = 1, \cdots, p,\ j = i+1, \cdots, p\ ]^T, \end{aligned} \quad (19)$$

where $a_{ij}$ is the $ij$th element of $\mathcal{A}_1$, $\bar{a}_{ij}$ and $\check{a}_{ij}$ are real and imaginary parts of $a_{ij}$, respectively, and $n$ is the number of unknown sensors. For $M > 1$ cases, the parameter set will be expanded to include elements of $\mathcal{A}_2, \mathcal{A}_3, \cdots, \mathcal{A}_M$.

4. Take the derivative of **R** with respect to the unknown parameters. This step is straightforward. After the results are substituted into (11), the CR bound matrix **CRBM** is obtained, and so is the MI-CR bound.

# 4 Numerical Examples

We now use examples to detail the procedure of the MI-CR bound calculation.

First, let us examine a line array of 8 sensors in Figure 1. This line array has two sensors of unkown response ($n = 2$). From the left to the right, if the first two sensors constitute the subset $S^1$, then the next 6 sensors form subsets $S_1^1$, $S_2^1$ and $S_3^1$, respectively. We denote the steering matrix corresponding to sensors in $S^1$ as

$$\mathcal{A}_1 = \begin{bmatrix} 1 & \cdots & 1 \\ \bar{a}_{21} + j\check{a}_{21} & \cdots & \bar{a}_{2p} + j\check{a}_{2p} \end{bmatrix}. \quad (20)$$

Then, the steering matrix **A** is given by $\mathbf{A}_1$ in (13) with $M = 1$, $q_1 = 3$, and all $\mathbf{J}_{ij}$ matrices are identity matrices. The complete parameter set is given by (19) with $n = 2$.

For the case of 200 snapshots and the correlation coefficient 0.99 of two emitter signals, we have calculated MI-CR bound when the DOAs are $[-7°, +7°]$. This bound, together with the simulation results using ESPRIT and the newly proposed algorithm — Weighted Subspace Estimation (WSE) [4, 5] are displayed in Figure 2. For the ESPRIT algorithm, the choice of the subarrays is shown in Figure 1. Figure 2 shows that, the performance of WSE achieves or is very close to the MI-CR bound.

Next, we consider a more complicated array configuration as shown in Figure 3 where 14 sensors are arranged in a block-wise uniform triangular form. The identical sensors are indicated by a common grayscale. The figure shows that there are 4 sensors whose responses are assumed to be unknown. Let $\mathcal{A}_1$ be the steering matrix for sensors 1 and 9, and $\mathcal{A}_2$ be the steering matrix for sensors 5 and 12, and let $\mathbf{\Phi}$ be the phase delay matrix corresponding to the displacement $\Delta = \lambda/2$. Then the steering matrix for the complete array has the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}, \quad (21)$$

$$\mathbf{A}_1 = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_1\mathbf{\Phi} \\ \mathcal{A}_1\mathbf{\Phi}^2 \\ \epsilon_1^T\mathcal{A}_1\mathbf{\Phi}^3 \end{bmatrix}, \quad (22)$$

$$\mathbf{A}_2 = \begin{bmatrix} \mathcal{A}_2 \\ \mathcal{A}_2\mathbf{\Phi} \\ \mathcal{A}_2\mathbf{\Phi}^2 \\ \epsilon_1^T\mathcal{A}_2\mathbf{\Phi}^3 \end{bmatrix}, \quad (23)$$

where $\epsilon_1$ is a unit vector with the first element being 1. After the derivatives of **R** in (5) are taken with respect to the unknown parameters and the result is substituted into (11), the CR bound can be calculated numerically, and the result is plotted in Figure 4. Also shown in this figure are the simulation results of ESPRIT and WSE. The subarray choice for the ESPRIT algorithm is shown in Figure 3. From Figure 4, we can see that the performance of ESPRIT is far from the MI-CR bound, while WSE is much closer to the bound.

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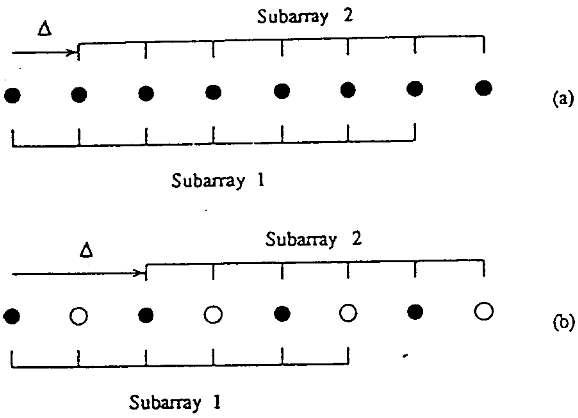



Figure 1: An example of line array.

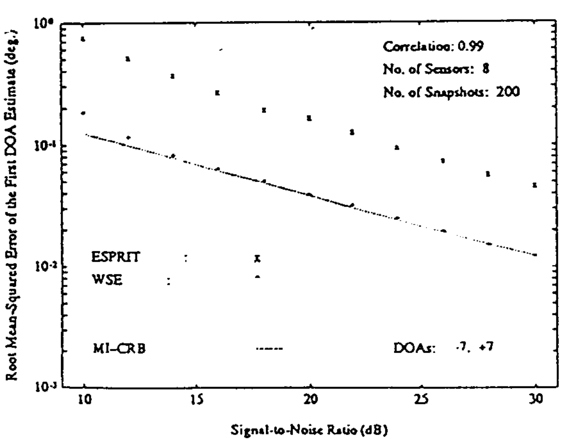
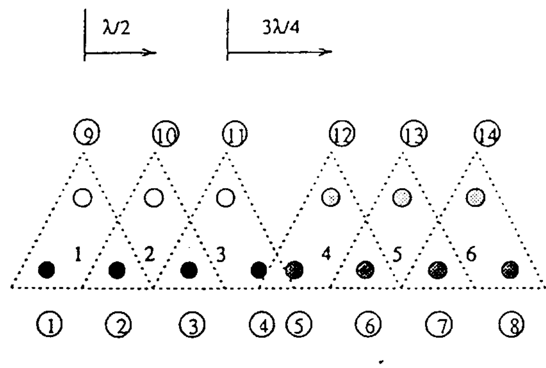



Figure 2: RMSE versus SNR for correlated sources




Subarray 1: triangles 1, 2, 4, 5;

Subarray 2: triangles 2, 3, 5, 6.

Figure 3: The geometry of block-wise uniform triangular array

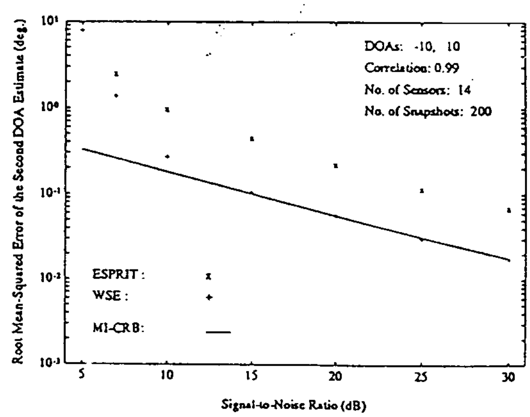



Figure 4: RMSE versus SNR for correlated sources