COMPARATIVE STUDIES OF MUSIC AND MVDR LOCATION ESTIMATORS FOR MODEL PERTURBATIONS

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ABSTRACT

A comparative study of the statistical performance of the MUSIC and minimum variance distortionless response (MVDR) direction of arrival (DOA) estimators is presented. Their relative performance due to given modeling errors is studied. In particular, we study the sensitivity of the estimators to modeling errors in the signal and noise structures.

1. INTRODUCTION

Direction of arrival (DOA) estimators include eigenspace based (e.g. MUSIC [1]) and spatial filter based (e.g. the minimum variance distortionless response (MVDR) [2]) techniques. A comprehensive comparative study of the statistical performance of the MUSIC and the MVDR estimators is important, but comparisons based on analytical expressions of estimator performance have been ignored in the literature to date. This paper provides an analytical study of the estimators. Our objectives are to provide a quantitative understanding of their relative performance due to given model errors, and to inspire further study. In particular, we will study the sensitivity of these estimators to modeling errors in the signal and noise structures.

Eigenspace based spectral methods have very high resolution properties and are based on the assumption that the signal observation belongs to a low rank subspace in spatially uncorrelated noise. However, these methods are sensitive to the white noise assumption and to incorrect model order selection. When these assumptions do not hold, DOA estimates will be biased and hence the methods will exhibit lower resolution. On the other hand, the MVDR method is not very sensitive to these assumptions. However, compared to eigenspace methods MVDR exhibits decreased resolution, at low SNR, under the additive white noise assumption [3]. The low resolution of MVDR is due to the fact that the DOA estimates are asymptotically biased[1].

The performance is compared for small random perturbations in the signal and noise models. The bias and standard deviation of the DOA estimate due to these random perturbations will provide the mean and the standard deviation, for particular realizations of the model errors, on the asymptotic bias obtained when processing under exact modeling assumptions. Relative finite data effects are considered as well. Since the asymptotic bias is related to the resolution properties of the estimators, we use its moments due to small perturbations for comparing the estimators. Not addressed here is the issue of model order selection of MUSIC. We use a signal subspace dimension equal to the number of sources. While in some perturbation cases MUSIC might perform better by selecting a different signal subspace dimension, generally the need for selecting order is an additional source of degradation.

2. BACKGROUND

Consider D narrow-band non-coherent signals radiating from source locations $\theta_1, \theta_2, ..., \theta_D$ impinging on an array of K sensors. The array response vector corresponding to each location θ_i is denoted as $\mathbf{a}(\theta_i)$. The covariance matrix $(K \times K)$ of the observation is

$$\mathbf{R} = \mathbf{A}\mathbf{S}\mathbf{A}^H + \sigma^2\mathbf{\Sigma}, \tag{1}$$

where $\mathbf{A} = [\mathbf{a}(\theta_1)\mathbf{a}(\theta_2), ..., \mathbf{a}(\theta_D)]$ is the $K \times D$ array response matrix; $\mathbf{S}$ and $\mathbf{\Sigma}$ are the signal and noise covariance matrices, respectively. $\mathbf{S}$ is assumed to be full rank (no coherent signals).

Assuming that the noise is spatially white ($\mathbf{\Sigma} = \mathbf{I}$), an eigendecomposition would give

$$\mathbf{R} = \mathbf{E}_s\mathbf{\Lambda}_s\mathbf{E}_s + \mathbf{E}_n\mathbf{\Lambda}_n\mathbf{E}_n, \tag{2}$$

where $\mathbf{E}_s = [\mathbf{e}_1, \cdots, \mathbf{e}_D]$ spans the signal subspace with $\mathbf{\Lambda}_s = diag(\lambda_1, \cdots, \lambda_D)$ and $\mathbf{E}_n = [\mathbf{e}_{D+1}, \cdots, \mathbf{e}_K]$ is orthogonal to the signal subspace with $\mathbf{\Lambda}_n = diag(\sigma^2, \cdots, \sigma^2)$. This property of orthogonality is exploited in MUSIC to form the null spectrum

$$f_{MU}(\theta) = \mathbf{a}^H(\theta)\mathbf{E}_n\mathbf{E}_n^H\mathbf{a}(\theta). \tag{3}$$

The DOA's are estimated from the minima of $f_{MU}(\theta)$. Finite data effects or small perturbations in the signal or noise model will perturb the null spectrum and lead to a bias in the estimates denoted by $\Delta_{MU}\theta_i$.

The MVDR spectrum estimator estimates the DOA's from the minima of the function

$$f_{MV}(\theta) = \mathbf{a}^H(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta). \tag{4}$$

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[1] Asymptotic implies that the number of snapshots tends to infinity.

Since the estimates are biased, the asymptotic location estimates are denoted by $\tilde{\theta}_i, i = 1, 2, \cdots, D$. Let

$$\Delta_{MV}\theta_i = (\tilde{\theta}_i - \theta_i) \quad (5)$$

denote the asymptotic bias, and define

$$\Delta_{MV}\tilde{\theta}_i = (\hat{\theta}_i - \tilde{\theta}_i) \quad (6)$$

to be the additional bias of the estimator.

To compare the performance of the estimators, the statistical properties of $\Delta_{MU}\theta_i$ and $\Delta_{MV}\tilde{\theta}_i$ have to be analyzed. Variance and bias expressions for $\Delta_{MU}\theta_i$ for eigenspace based spectrum estimation methods have been derived under different settings in [4], [5], [6]. Performance of the MVDR method, under finite data effects and model perturbations in the signal and noise models, has been studied in [7]. With the aid of the analytical expressions derived in the references mentioned above, we will present comparative study of the variance and bias of these estimators under different scenarios.

3. COMPARATIVE STUDY

For the study presented the following scenario is assumed. Two sources with equal signal strengths at 10^o and 20^o are considered. A 10 element uniform linear array with half wavelength spacing is chosen. The noise is assumed to be white. The array response vectors are all normalized to one and the signal to noise ratio (SNR) is computed as $10 \log \frac{\mathbf{S}_{ii}}{\sigma^2}$.

3.1. Performance criteria

In the case of MUSIC, if the noise is not spatially white (and assuming no knowledge of the noise covariance structure) or if the model order is not determined exactly, the asymptotic estimates would be biased. We would then get two terms contributing to the bias, i.e. $\Delta_{MU}\theta_i$ and $\Delta_{MU}\tilde{\theta}_i$. To exploit the results in [4], [6], the comparison results presented here are for the case when the noise is assumed spatially white.

For small signal and noise model perturbations, the performance of these estimators has been studied [6], [7]. In practice, we might have one realization of these errors and thus the moments of the estimates will provide the mean and the standard deviation, for a particular realization of the model errors, on the asymptotic bias. We compare the standard deviation of these estimators and the asymptotic bias of the MVDR estimator due to different modeling errors. Note that the statistical or perturbation bias term for MUSIC and the additional (statistical) bias term for MVDR is also an important measure in some situations, but they are not considered here. Effects of finite data lengths are also analyzed. For a particular example of correlated noise structure, we will compare the asymptotic biases.

3.2. Finite data effects

The variance expression for MUSIC is given by[2] [4]

$$\mathcal{E}(\Delta_{MU}\theta_i)^2 \simeq \frac{1}{N} \frac{\sum_{k=1}^{D} \frac{\lambda_k \sigma^2}{(\lambda_k - \sigma^2)} |\mathbf{a}^H(\theta_i)\mathbf{e}_k|^2}{\ddot{f}_{MU}(\theta_i)}. \quad (7)$$

[2] $\simeq$ denotes an approximation.

For the MVDR spectrum estimator the variance of the DOA estimates are given by [7]

$$\mathcal{E}(\Delta_{MV}\tilde{\theta}_i)^2 \simeq \frac{2(N-K)Re[\mathbf{a}^H(\tilde{\theta}_i)\mathbf{R}^{-1}\mathbf{B}(\tilde{\theta}_i)\mathbf{R}^{-1}\dot{\mathbf{a}}(\tilde{\theta}_i)]}{(N-K-1)(N-K+1)\ddot{f}^2_{MV}(\tilde{\theta}_i)} \quad (8)$$

where

$$\mathbf{B}(\tilde{\theta}_i) = \dot{\mathbf{a}}(\tilde{\theta}_i)\mathbf{a}^H(\tilde{\theta}_i) + \mathbf{a}(\tilde{\theta}_i)\dot{\mathbf{a}}^H(\tilde{\theta}_i). \quad (9)$$

In the above expressions, $\dot{v}$ and $\ddot{v}$ denote the first and second derivative of the function $v(\theta)$ with respect to θ, respectively.

Figure 1 shows the standard deviation for the two estimators. The number of snapshots are assumed to be 50. The dotted line shows the asymptotic bias of the MVDR estimator. For large SNR's, the standard deviation is the dominant term and the performance is similar for both the estimators. At low SNR's, the asymptotic bias limits the MVDR estimator. The asymptotic bias also limits the MVDR for sources spaced closer at high SNR's.

We next study the effects of modeling errors on the performance of the estimators. Two cases are considered independently; the effects of randomly perturbing the array response model and randomly perturbing the noise covariance structure.

3.3. Array model perturbations

For the array response vector case, the errors are taken to be additive to the actual response matrix $\mathbf{A}$, i.e.

$$\hat{\mathbf{A}} = \mathbf{A} + \Delta\mathbf{A}. \quad (10)$$

The performance is studied for the case where the perturbations are random, zero mean, with the following second order moments[3]:

$$\mathcal{E}[\Delta\mathbf{a}(\theta_i)\Delta\mathbf{a}^H(\theta_j)] = \delta_{ij}\mathbf{\Psi} \quad (11)$$

$$\mathcal{E}[\Delta\mathbf{a}(\theta_i)\Delta\mathbf{a}^T(\theta_j)] = 0. \quad (12)$$

The variance for MUSIC is given by [6]

$$\mathcal{E}(\Delta_{MU}\theta_i^2) \simeq \frac{\sigma_a^2}{2} \frac{1}{\dot{\mathbf{a}}^H(\theta_i)\mathbf{E}_n\mathbf{E}_n^H\dot{\mathbf{a}}(\theta_i)}. \quad (13)$$

Define

$$\mathbf{\Phi} = \mathbf{A}\mathbf{S}^2\mathbf{A}^H. \quad (14)$$

For the MVDR estimator the variance is given by [7]

$$\mathcal{E}(\Delta_{MV}\tilde{\theta}_i^2) \simeq \frac{2}{\ddot{f}^2_{MV}(\tilde{\theta}_i)} \; Re[\mathbf{a}^H(\tilde{\theta}_i)\mathbf{\Phi}^R\mathbf{B}(\tilde{\theta}_i)\mathbf{\Psi}^R\dot{\mathbf{a}}(\tilde{\theta}_i) + \mathbf{a}^H(\tilde{\theta}_i)\mathbf{\Psi}^R\mathbf{B}(\tilde{\theta}_i)\mathbf{\Phi}^R\dot{\mathbf{a}}(\tilde{\theta}_i)], \quad (15)$$

where

$$\mathbf{\Phi}^R = \mathbf{R}^{-1}\mathbf{\Phi}\mathbf{R}^{-1} \quad \& \quad \mathbf{\Psi}^R = \mathbf{R}^{-1}\mathbf{\Psi}\mathbf{R}^{-1}. \quad (16)$$

Figure 2 shows the the standard deviation for the two estimators. For this example, we choose $\mathbf{\Psi} = \sigma_a^2\mathbf{I}$. The performance is shown as a function of the signal to perturbation ratio (defined as $10 \log \frac{\mathbf{S}_{ii}}{\sigma_a^2}$). The SNR is fixed

[3] See [6] for a good discussion on this model.

at $30dB$ (the performance was independent of SNR). The standard deviations of the estimators are identical in this case. The contribution to the asymptotic bias due to model perturbation dominates the exact model asymptotic bias of MVDR. To illustrate the behavior of the spectra of the two estimators under ideal modeling conditions and for different realizations of the model perturbations, example spectra for different realizations of the model perturbation are shown in figures 4 and 5. Observe that the spectra is peakier for MUSIC under ideal conditions. However, the spectra are almost identical for the two estimators even for small model perturbations. The signal to perturbation ratio was taken to be $30dB$; a very small amount perturbation can lead to rapid degradation of the MUSIC spectrum. In this general situation, and others we have considered, the performance of the estimators is similar for different values of SNR and SPR up to the resolution threshold. For closely spaced sources, small perturbations in the signal model rapidly degrade the high resolution characteristics of the estimators.

3.4. Noise covariance perturbations

The perturbations on the noise covariance matrix are assumed zero mean with the following second order moments:

$$\mathcal{E}[\Delta\Sigma_{ij}\Delta\Sigma_{kl}] = \mu^2\delta_{il}\delta_{kj}. \tag{17}$$

The variance expression for MUSIC is given by [6]

$$\mathcal{E}(\Delta_{MU}\tilde{\theta}_i)^2 \simeq \frac{\mu^2\sigma^4}{2}\frac{Re[(\mathbf{S}\mathbf{A}^H(\theta)\mathbf{A}(\theta)\mathbf{S})^{-1}]_{ii}}{\bar{f}_{MU}(\theta_i)}. \tag{18}$$

Similarly, for MVDR the variance expression is [7]

$$\mathcal{E}(\Delta_{MV}\tilde{\theta}_i)^2 \simeq \frac{2\mu^2}{\bar{f}^2_{MV}(\tilde{\theta}_i)} Re(\mathbf{a}^H(\tilde{\theta}_i)\mathbf{R}^{-2}\mathbf{B}(\tilde{\theta}_i)\mathbf{R}^{-2}\dot{\mathbf{a}}(\tilde{\theta}_i)). \tag{19}$$

Figure 3 shows the plot for the standard deviation of the two estimators as function of the signal to perturbation ratio. The SNR was varied from $10dB$ to $30dB$ to $50dB$. The signal to perturbation ratio is calculated as $10\log\frac{\mu^2}{\sigma^2}$. For this case neither MVDR nor MUSIC degrade significantly. So the exact model asymptotic bias of MVDR is the dominate factor in comparing MUSIC and MVDR performance. This case illustrates that small noise perturbations do not significantly affect the signal subspace, and hence the MUSIC performs significantly better than MVDR.

3.5. Asymptotic bias

In the preceding analyses, the noise was assumed white and, therefore, MUSIC had no exact model asymptotic bias. However, when the noise is correlated and no knowledge of the noise covariance structure is assumed, MUSIC will exhibit asymptotic bias. Since this situation will often occur in practice it is interesting to compare the asymptotic bias of the estimators as a function of noise coherence. The noise model is assumed to be of the form [3]

$$\Sigma_{kl} = \rho^{|l-k|}. \tag{20}$$

This model generates a spatially distributed source centered around DOA $0°$. Figure 6 shows the behavior of the absolute value of the asymptotic bias as function of the noise coherence factor ρ. Note that $\rho = 0$ corresponds to white noise. In this example, observe that the asymptotic bias of MUSIC is significant and increases with ρ.

4. CONCLUSIONS

An analytical comparison of the MUSIC and MVDR DOA estimators is performed. The variances of the two estimators due to small random perturbations in the array response vector model are affectively identical, as are the the variances due to finite sample effects. The MVDR spectrum estimator has an exact model asymptotic bias. However we've illustrated that standard deviation, due to small perturbations in the signal model and certain perturbations in the noise model, can dominate this asymptotic bias. If this is the case, MUSIC and MVDR will perform similarly.

5. REFERENCES

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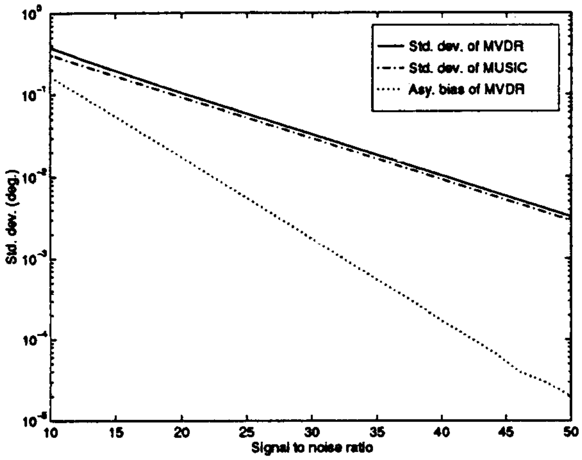

Figure 1: Standard deviation for finite sample effects. $30dB$ SN

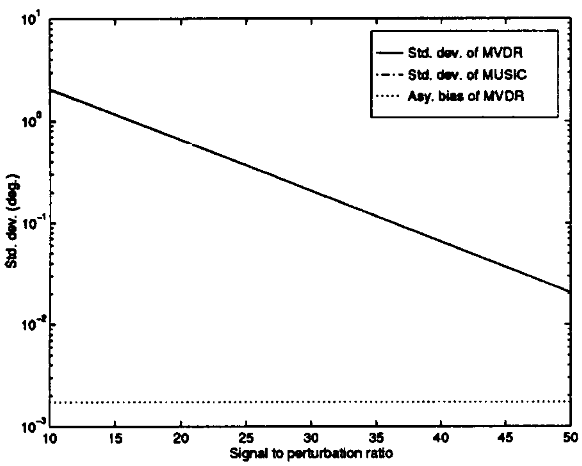

Figure 2: Standard deviation versus perturbation in array res

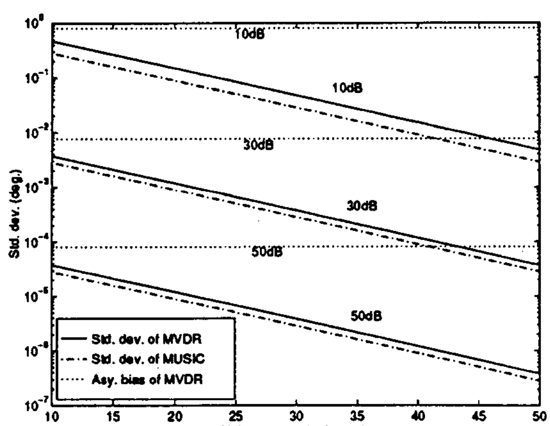

Figure 3: Standard deviation versus perturbation in noise covariance. Source at 10°.

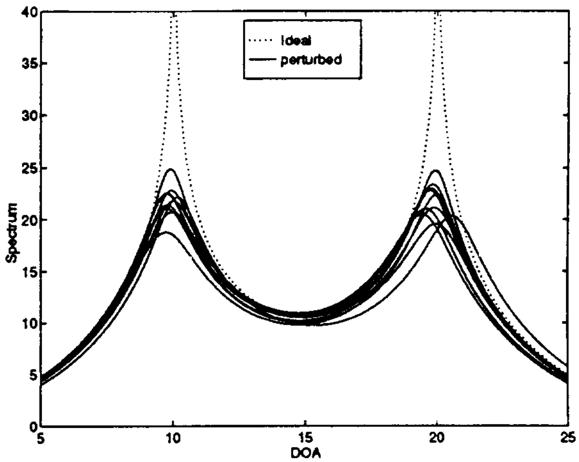

Figure 4: MUSIC spectrum. $30dB$ SNR and $30dB$ SPR.

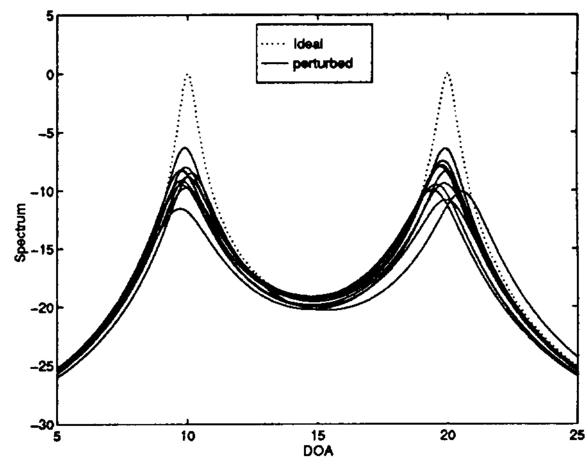

Figure 5: MVDR spectrum. $30dB$ SNR and $30dB$ SPR.

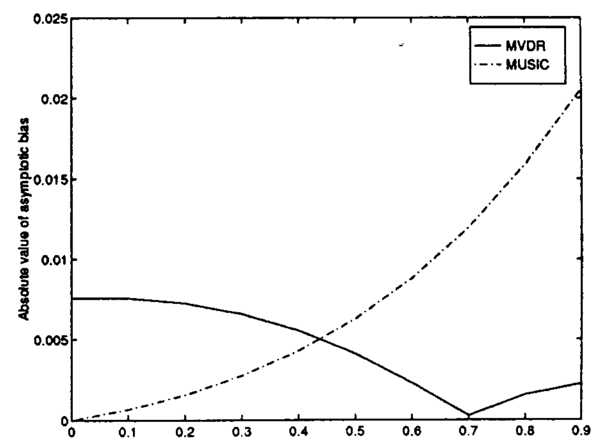

Figure 6: Absolute value of the asymptotic bias. $30dB$ SNR. Source at 10°.