PERFORMANCE ANALYSIS OF THE MINIMUM VARIANCE BEAMFORMER

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ABSTRACT

We present an analysis of the Signal-to-Interference-plus-Noise Ratio (SINR) at the output of the Minimum Variance beamformer. The analysis yields an explicit expression for the SINR in terms of the different parameters affecting the performance, including the Signal-to-Noise Ratio (SNR), the Interference-to-Noise Ratio (INR), the Signal-to-Interference Ratio (SIR), the angular separation between the desired signal and the interference, the array size and shape, the correlation between the desired signal and the interference, and the finite sample size.

I. INTRODUCTION

The Minimum Variance beamformer is an important and popular adaptive beamforming technique. Yet, in spite of its popularity and long history its performance has been analyzed only partially.

The parameter of interest in the beamformer performance is the ratio of desired signal power to the interference-plus-noise power, referred to as the signal-to-interference-plus-noise ratio (SINR). This ratio is affected by many parameters including the Signal-to-Noise Ratio (SNR), the Interference-to-Noise Ratio (INR), the Signal-to-Interference Ratio (SIR), the angular separation between the desired signal and the interference, the array size and shape, the correlation between the desired signal and the interference, and the finite sample size.

No complete analysis of the SINR as a function of *all* these parameters has been presented. All the works [1]-[7] were confined to the analysis of different subsets of these parameters.

In this paper we present a complete analysis of the SINR as a function of all the above mentioned parameters. The analysis is based on a novel expression for the weight vector, and on a novel definition of the SINR introduced to handle also the case of signal cancellation, occurring when the desired signal and the interference are correlated.

II. PROBLEM FORMULATION

Suppose it is desired to receive only the source at θ_1, referred to as the desired source, and reject all the other $q-1$ sources, referred to as interferences.

Using complex envelope representation, the $p \times 1$ vector received by the array can be expressed as

$$\mathbf{x}(t) = \mathbf{a}(\theta_1)s_1(t) + \mathbf{v}(t), \tag{1}$$

where $\mathbf{a}(\theta)$ denotes the $p \times 1$ "steering vector" of the array towards direction θ, $\mathbf{v}(t)$ denotes the interference-plus-noise vector

$$\mathbf{v}(t) = \sum_{k=2}^{q} \mathbf{a}(\theta_k)s_k(t) + \mathbf{n}(t), \tag{2}$$

and $s_k(t)$ denotes the signal of the k-th source as received at the reference point.

The following statistical models for the noise and signals are assumed:

A1: The noise samples $\{\mathbf{n}(t_i)\}$ are i.i.d. Gaussian random vectors with zero mean and covariance matrix $\sigma_n^2\mathbf{I}$, where $\mathbf{I}$ denotes the $p \times p$ identity matrix.

A2: The signal samples $\{\mathbf{s}(t_i) = [s_1(t_i), \ldots, s_q(t_i)]^T\}$ are i.i.d Gaussian random vectors, independent of the noise samples, with zero mean and arbitrary $q \times q$ covariance matrix $\mathbf{C}$.

Notice that the correlation amongst the signals can be arbitrary. Specifically, the desired signal can be fully correlated with the interferences, as happens in the case of specular multipath propagation and "smart" jamming.

Assuming that the direction of the desired signal, θ_1, is *known*, the output of the Minimum Variance beamformer is given by

$$\hat{s}_1(t) = \mathbf{w}^H\mathbf{x}(t), \tag{3}$$

where $\mathbf{w}$ is the adaptive weight vector given by the well known expression

$$\mathbf{w} = \frac{1}{\mathbf{a}_1^H\hat{\mathbf{R}}^{-1}\mathbf{a}_1}\hat{\mathbf{R}}^{-1}\mathbf{a}_1, \tag{4}$$

with $\mathbf{a}_1$ standing for $\mathbf{a}(\theta_1)$ and $\hat{\mathbf{R}}$ denoting the sample-covariance matrix computed from m samples of $\mathbf{x}(t)$ at

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time instants $t_1, \ldots, t_m$,

$$\hat{\mathbf{R}} = \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}(t_i)\mathbf{x}^H(t_i). \quad (5)$$

The problem is to derive and analyzed the ratio of the desired signal power to the interference-plus-noise power, namely the Signal-to-Interference-plus-Noise Ratio (SINR), at the beamformer output.

III. THE WEIGHT VECTOR

In [8] we show that the MV weight vector (4) can be expresed as

$$\mathbf{w} = \frac{1}{\mathbf{a}_1^H\hat{\mathbf{Q}}^{-1}\mathbf{a}_1}\hat{\mathbf{Q}}^{-1}\mathbf{a}_1 - \left[\mathbf{I} - \frac{\hat{\mathbf{Q}}^{-1}\mathbf{a}_1\mathbf{a}_1^H}{\mathbf{a}_1^H\hat{\mathbf{Q}}^{-1}\mathbf{a}_1}\right]\hat{\mathbf{Q}}^{-1}\hat{\mathbf{r}}, \quad (6)$$

where $\hat{\mathbf{r}}$ denotes the sample-mean of the correlation between the desired signal and the interference-plus-noise,

$$\hat{\mathbf{r}} = \frac{1}{m}\sum_{i=1}^{m}s_1^*(t_i)\mathbf{v}(t_i), \quad (7)$$

and $\hat{\mathbf{Q}}$ denotes the sample-covariance of the interference-plus-noise,

$$\hat{\mathbf{Q}} = \frac{1}{m}\sum_{i=1}^{m}\mathbf{v}(t_i)\mathbf{v}^H(t_i). \quad (8)$$

This expression displays clearly the decomposition of $\mathbf{w}$ into the desired value and the undesired perturbation, caused by the finite sample-size and the correlation between the desired signal and the interferences. Indeed, the first term is the *ideal* weight vector, i.e, the one that maximizes the signal-to-interference-plus-noise-ratio, while the second term can be regarded as a *perturbation*, resulting from the sample-correlation $\hat{\mathbf{r}}$ between the desired signal and the interference-plus-noise. Notice that because of the *finite sample-size*, this sample-correlation is *nonzero*, with probability one, even if the interference is uncorrelated with the desired signal.

To facilitate the analysis we approximate $\mathbf{w}$ by modifying (6) to the following expression,

$$\mathbf{w} \simeq \frac{1}{\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{a}_1}\mathbf{Q}^{-1}\mathbf{a}_1 - \mathbf{P}\mathbf{Q}^{-1}\hat{\mathbf{r}}, \quad (9)$$

where we have replaced the sample-covariance $\hat{\mathbf{Q}}$ by the exact covariance

$$\mathbf{Q} = E[\mathbf{v}(t)\mathbf{v}^H(t)], \quad (10)$$

and denoted by $\mathbf{P}$ the oblique projection

$$\mathbf{P} = \mathbf{I} - \frac{\mathbf{Q}^{-1}\mathbf{a}_1\mathbf{a}_1^H}{\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{a}_1}. \quad (11)$$

Though this approximation is rather poor for very low sample-size, *i.e.*, for $m \approx p$, it is valid for large sample size and even for moderate sample size, *i.e.*, for $m > 3p$, and turns out to have a very marginal effect on the SINR. Indeed, from the analysis of Reed *et al.* [67], who considered the case that the desired signal is absebt from $\hat{\mathbf{R}}$ and hence $\mathbf{w}$ is given by the first term of (6), when $m \geq 3p$ the effect of replacing $\hat{\mathbf{Q}}$ with $\mathbf{Q}$ in the first term of (6) causes an error of about $1.5dB$ in the SINR. The effect on the SINR of replacing $\hat{\mathbf{Q}}$ by $\mathbf{Q}$ in the second term of (6) is also marginal since the sample-correlation $\hat{\mathbf{r}}$ still captures most the finite sample-size effect.

IV. GENERAL EXPRESSION FOR SINR

Consider the Hilbert space with the inner product $(u(t), v(t)) = E[u(t)v^*(t)]$, where * denotes the complex conjugate, and $E[\cdot]$ denotes the expectation operator. In this space we can decompose $\hat{s}_1(t)$ into the following orthogonal decomposition

$$\hat{s}_1(t) = ks_1(t) + \tilde{n}_1(t), \quad (12)$$

where k is a complex scalar and $\tilde{n}_1(t)$ is orthogonal to $s_1(t)$, *i.e.*,

$$E[s_1(t)\tilde{n}_1^*(t)] = 0. \quad (13)$$

Multiplying (12) by $s_1^*(t)$, taking expectations and using (13), we get

$$k = \frac{E[\hat{s}_1(t)s_1^*(t)]}{E[|s_1(t)|^2]}. \quad (14)$$

That is, k is essentially the correlation coefficient between the beamformer output and the desired signal.

The SINR at the beamformer output can be readily obtained from (12). Indeed, since the signal power is given by $E[|ks_1(t)|^2]$, while the noise power is given by $E[|\hat{s}_1(t) - ks_1(t)|^2]$, we get

$$SINR = \frac{E[|ks_1(t)|^2]}{E[|\hat{s}_1(t) - ks_1(t)|^2]}, \quad (15)$$

which by inserting (14) yields

$$SINR = \frac{|E[\hat{s}_1(t)s_1^*(t)]|^2}{E[|\hat{s}_1(t)|^2]E[|s_1(t)|^2] - |E[\hat{s}_1(t)s_1^*(t)]|^2}. \quad (16)$$

Carrying out the expectations in (16), using (3) and (9), we get [8]

$$SINR = \frac{S}{I+N}, \quad (17)$$

where S denotes the desired signal power

$$\begin{aligned} S \simeq\ & \sigma_{s_1}^2 + 2\mathrm{Re}\{\frac{\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{r}}{\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{a}_1}\} - 2\mathbf{r}^H\mathbf{P}\mathbf{Q}^{-1}\mathbf{r} \\ & + \frac{1}{\sigma_{s_1}^2}\frac{|\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{r}|^2}{(\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{a}_1)^2} - 2\frac{1}{\sigma_{s_1}^2}\frac{\mathbf{r}^H\mathbf{P}\mathbf{Q}^{-1}\mathbf{r}}{\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{a}_1}\mathrm{Re}\{\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{r}\}, \\ & + \frac{1}{\sigma_{s_1}^2}|\mathbf{r}^H\mathbf{P}\mathbf{Q}^{-1}\mathbf{r}|^2, \end{aligned} \quad (18)$$

while $I+N$ denotes the interference-plus-noise power

$$I+N \simeq \frac{1}{\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{a}_1} + \frac{p-1}{m}\sigma_{s_1}^2 + \mathbf{r}^H\mathbf{P}\mathbf{Q}^{-1}\mathbf{r} - \frac{1}{\sigma_{s_1}^2}\frac{|\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{r}|^2}{(\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{a}_1)^2} + 2\frac{1}{\sigma_{s_1}^2}\frac{\mathbf{r}^H\mathbf{P}\mathbf{Q}^{-1}\mathbf{r}}{\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{a}_1}\mathrm{Re}\{\mathbf{a}_1^H\mathbf{Q}^{-1}\mathbf{r}\} - \frac{1}{\sigma_{s_1}^2}|\mathbf{r}^H\mathbf{P}\mathbf{Q}^{-1}\mathbf{r}|^2, \quad (19)$$

where $\mathbf{r} = E[s_1^*(t)\mathbf{v}(t)]$ and $\sigma_{s_1}^2 = E[|s_1(t)|^2]$.

V. SINGLE INTERFERENCE CASE

In the case of a single interference the SINR reduces to

$$SINR = \frac{\gamma}{\delta}, \quad (20)$$

where

$$\gamma \simeq |\mathbf{a}_1|^2[1 - \frac{INR|\mathbf{a}_2|^2|\rho|^2(1-|\alpha|^2)}{INR|\mathbf{a}_2|^2(1-|\alpha|^2)+1}]^2 + \frac{|\mathbf{a}_2|^2|\rho|^2|\alpha|^2}{[INR|\mathbf{a}_2|^2(1-|\alpha|^2)+1]^2 SIR} + \frac{2|\mathbf{a}_1||\mathbf{a}_2|[(INR|\mathbf{a}_2|^2(1-|\alpha|^2)(1-|\rho|^2)+1]\mathrm{Re}\{\rho\alpha\}}{\sqrt{SIR}[INR|\mathbf{a}_2|^2(1-|\alpha|^2)+1]^2}, \quad (21)$$

and

$$\delta \simeq \frac{INR|\mathbf{a}_2|^2+1}{SNR[INR|\mathbf{a}_2|^2(1-|\alpha|^2)+1]} + |\mathbf{a}_1|^2\frac{p-1}{m} + \frac{INR|\mathbf{a}_1|^2|\mathbf{a}_2|^2|\rho|^2(1-|\alpha|^2)}{INR|\mathbf{a}_2|^2(1-|\alpha|^2)+1}\cdot(1 - \frac{INR|\mathbf{a}_2|^2|\rho|^2(1-|\alpha|^2)}{INR|\mathbf{a}_2|^2(1-|\alpha|^2)+1}) - \frac{|\mathbf{a}_2|^2|\rho|^2|\alpha|^2}{SIR[INR|\mathbf{a}_2|^2(1-|\alpha|^2)+1]^2} + \frac{2INR|\mathbf{a}_1||\mathbf{a}_2|^3|\rho|^2(1-|\alpha|^2)\mathrm{Re}\{\rho\alpha\}}{\sqrt{SIR}[INR|\mathbf{a}_2|^2(1-|\alpha|^2)+1]^2}, \quad (22)$$

where α denotes the spatial correlation

$$\alpha = \frac{\mathbf{a}_1^H\mathbf{a}_2}{|\mathbf{a}_1||\mathbf{a}_2|}, \quad (23)$$

and ρ denotes the correlation coefficient between the desired signal and the interference

$$\rho = \frac{E[s_1^*(t)s_2(t)]}{\sigma_{s_1}\sigma_{s_2}}. \quad (24)$$

This expression allows to readily compute the SINR for any given array, any number of samples m, and any set of parameters $\{SNR, INR, SIR, |\mathbf{a}_1|, |\mathbf{a}_2|, |\alpha|, |\rho|\}$, characterizing the desired signal and the interference.

We next present an asymptotical analysis of this expression in two special cases wherein it simplifies considerably and allows to gain insight into the SINR behaviour.

A. UNCORRELATED INTERFERENCE

In case the interference and the desired signal are uncorrelated (i.e., $\rho = 0$) considerable simplification arise for high INR obeying $INR \gg \frac{1}{(1-|\alpha|^2)|\mathbf{a}_2|^2}$.In this case (20)-(22) become

$$SINR \simeq \frac{SNR|\mathbf{a}_1|^2}{\frac{1}{1-|\alpha|^2} + SNR|\mathbf{a}_1|^2\frac{p-1}{m}} \quad (25)$$

Thus, for $SNR \ll \frac{m}{(p-1)(1-|\alpha|^2)|\mathbf{a}_1|^2}$, we have

$$SINR \simeq SNR|\mathbf{a}_1|^2(1-|\alpha|^2), \quad (26)$$

while for $SNR \gg \frac{m}{(p-1)(1-|\alpha|^2)|\mathbf{a}_1|^2}$, we have

$$SINR \simeq \frac{m}{p-1}. \quad (27)$$

That is, the SINR increases linearly with the SNR, with a slope determined by the spatial correlation between the steering vectors; the more correlated these vectors are, i.e., the larger is $|\alpha|$, the more moderate is the slope. For high SNR the SINR levels up at $\frac{m}{p-1}$.

B. CORRELATED INTERFERENCE

In case the interference is correlated with the desired signal, considerable simplifications arise for high INR obeying $INR \gg \frac{1}{(1-|\alpha|^2)|\mathbf{a}_2|^2}$. In this case (20)-(22) become

$$SINR \simeq \frac{SNR|\mathbf{a}_1|^2(1-|\rho|^2)^2}{\frac{1}{(1-|\alpha|^2)} + SNR|\mathbf{a}_1|^2[|\rho|^2(1-|\rho|^2)+\frac{p-1}{m}]}. \quad (28)$$

Thus, for $SNR \ll \frac{1}{(|\rho|^2(1-|\rho|^2)+\frac{p-1}{m})(1-|\alpha|^2)|\mathbf{a}_1|^2}$, we get

$$SINR \simeq SNR|\mathbf{a}_1|^2(1-|\rho|^2)^2(1-|\alpha|^2), \quad (29)$$

while for $SNR \gg \frac{1}{(|\rho|^2(1-|\rho|^2)+\frac{p-1}{m})(1-|\alpha|^2)|\mathbf{a}_1|^2}$, we get

$$SINR \simeq \frac{(1-|\rho|^2)^2}{\frac{p-1}{m} + |\rho|^2(1-|\rho|^2)}. \quad (30)$$

That is, as in the case of uncorrelated sources, the maximum SINR is limited. Yet, the limit in this case is determined primarily by the correlation coefficient ρ between the desired signal and the interference.

VI. SIMULATION RESULTS

To confirm the analysis and gain more insight into the achievable performance, we present results of simulated experiments and compare them with the analytical results. The results of the simulations were computed from the sample-average of 100 Monte-Carlo runs, with each run consisting of 1000 samples of $\mathbf{x}(t)$.

The array was a 5 element uniform circular array with diameter $d = 0.8\lambda$, where λ denotes the wavelength. The impinging sources consisted of a desired signal and a single interference. The angular separation between them was a parameter with three values: $0.2\frac{d}{\lambda}$, $0.5\frac{d}{\lambda}$ and $1.0\frac{d}{\lambda}$, which for the given array correspond to spatial correlation values $|\alpha| = 0.96$, $|\alpha| = 0.77$ and $|\alpha| = 0.25$, respectively.

In the first experiment the interference was *uncorrelated* with the desired signal, i.e., $\rho = 0$, the SIR was held fixed at $SIR = 20db$, and the number of samples was $m = 1000$. The results, presented in Figure 1, show that the SINR rises linearly with the SNR and then levels up at $\frac{m}{p-1} = \frac{1000}{4} = 24db$, as predicted by the analysis.

In the second experiment the scenario was as in the first experiment except that the interference was *correlated* with the desired signal, with the correlation coefficient being $\rho = 0.4$. The results, presented in Figure 2, show that the SINR rises linearly and identically for the three angular separations, reaches a maximum and then decreases to the value $\frac{(1-|\rho|^2)^2}{|\rho|^2(1-|\rho|^2)+\frac{p-1}{m}} = \frac{(1-0.4^2)^2}{0.4^2(1-0.4^2)+\frac{4}{1000}} = 7.1db$, as predicted by the analysis.

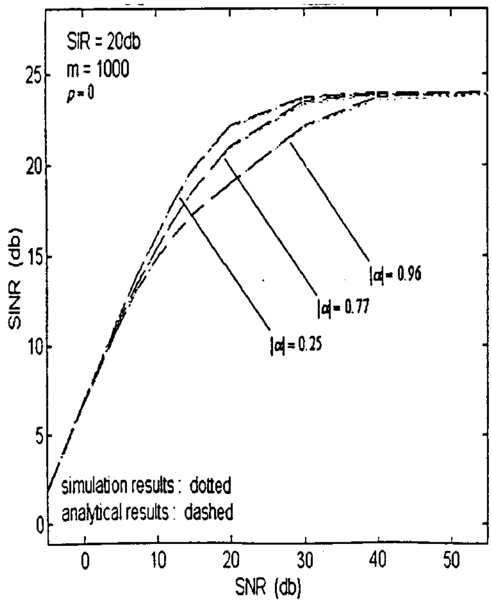

Figure 1

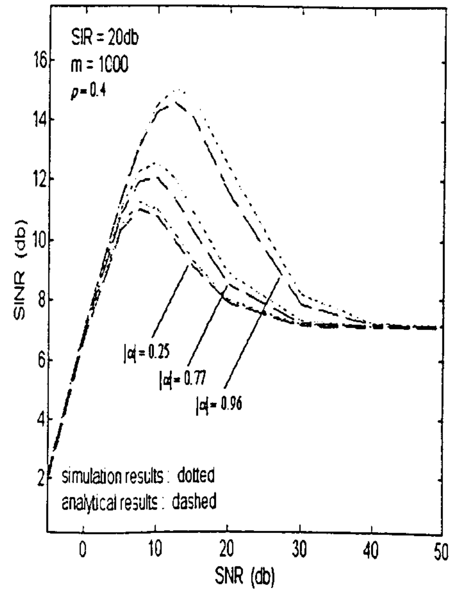

Figure 2

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