ANALYSING THE EFFECTS OF CONSTRAINTS AND INTER-SIGNAL COHERENCE ON THE MUSIC ALGORITHM

D. Linebarger, R. DeGroat, E. Dowling, G. Fudge

The University of Texas at Dallas
Richardson, TX 75083-0688
dl@utdallas.edu

Petre Stoica

Dept. of Applied Electronics
Chalmers University of Technology
S-412 96 Gothenburg, Sweden

ABSTRACT

We perform analysis of constrained and unconstrained MUSIC demonstrating that (asymptotically) improved subspace estimates *always* result from the use of constraints, and (asymptotically) the variance of constrained MUSIC is less than that of unconstrained MUSIC under either high coherence, large numbers of sensors, or high SNR conditions. As part of this analysis, we study the effects of coherence on MUSIC and derive best/worst case coherences in terms of the variance of MUSIC. We also demonstrate that those conditions where the variance of MUSIC is predicted to be less than that of constrained MUSIC generally correspond to conditions where MUSIC is in breakdown (and constrained MUSIC is not). So, unconstrained MUSIC does not achieve its predicted advantage in those cases.

1. INTRODUCTION

This paper follows a sequence of papers that dealt with variations of MUSIC [1] involving different types of *a priori* information regarding signal directions. The variations of MUSIC include constrained MUSIC (alluded to in [2] and formally defined in [3]), beamspace MUSIC [4], and constrained beamspace MUSIC [5]. Papers containing analysis of these algorithms include this one as well as [6, 7, 5, 8]. The material in this paper appears in expanded form in [8], which is available by email request to dl@utdallas.edu.

We assume the following correlation matrix model:

$$\begin{aligned}\mathbf{R} &= \mathcal{E}[\mathbf{x}_k\mathbf{x}_k^H] \\ &= \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I}. \quad (1)\\ &= [\mathbf{V}_s \quad \mathbf{V}_n]\begin{bmatrix}\mathbf{\Lambda}_s & 0\\ 0 & \sigma^2\mathbf{I}\end{bmatrix}[\mathbf{V}_s \quad \mathbf{V}_n]^H \quad (2)\end{aligned}$$

The spectral MUSIC estimator involves evaluation of

$$\min_{\theta} \qquad f(\theta) = \mathbf{a}^H(\theta)\,\widehat{\mathbf{V}}_n\widehat{\mathbf{V}}_n^H\mathbf{a}(\theta). \qquad (3)$$

The matrix $\widehat{\mathbf{V}}_n$ corresponds to the eigenvectors spanning the noise subspace of $\widehat{\mathbf{R}}$, the estimated correlation matrix. Constrained, beamspace, and constrained beamspace versions of MUSIC have a similar form, but they involve eigendecompositions of transformed versions of the data. If a transformed estimated correlation matrix is defined according to

$$\widehat{\mathbf{S}} = \mathbf{Q}^H\widehat{\mathbf{R}}\mathbf{Q} = [\widehat{\mathbf{U}}_s \quad \widehat{\mathbf{U}}_n]\begin{bmatrix}\widehat{\mathbf{D}}_s & \mathbf{0}\\ \mathbf{0} & \widehat{\mathbf{D}}_n\end{bmatrix}[\widehat{\mathbf{U}}_s \quad \widehat{\mathbf{U}}_n]^H. \quad (4)$$

then spectral MUSIC for the transformed data is

$$\min_{\theta} \qquad f(\theta) = \mathbf{a}^H(\theta)\mathbf{Q}\widehat{\mathbf{U}}_n\widehat{\mathbf{U}}_n^H\mathbf{Q}^H\mathbf{a}(\theta). \qquad (5)$$

where $\mathbf{Q}$ is different for each variation of MUSIC [5].

2. CONSTRAINTS = IMPROVED SUBSPACES

In this section, we analyze the subspaces resulting from the EVD of the original correlation matrix and compare them to subspaces obtained from an EVD of the constrained (transformed) correlation matrix. Based on the eigenvector covariance relations in [9], we derived the subspace distance (SD) (using the Frobenius norm) between the estimated and actual signal subspaces as (derivation given in [8]):

$$\begin{aligned}\mathrm{SD} &= \left\|\mathbf{V}_s\mathbf{V}_s^H - \widehat{\mathbf{V}}_s\widehat{\mathbf{V}}_s^H\right\|_F^2 \\ &= \left\|\mathbf{V}_s\mathbf{V}_s^H - (\mathbf{V}_s+\mathbf{E})(\mathbf{V}_s+\mathbf{E})^H\right\|_F^2 \\ &\approx \frac{2\sigma^2(m-q)}{N}\sum_{i=1}^{q}\frac{\lambda_{s,i}}{(\lambda_{s,i}-\sigma^2)^2} \qquad (6)\end{aligned}$$

(The matrix $\mathbf{E}$ represents the error in the estimated eigenvectors.) The next issue we must determine is whether the subspace distance (as specified above) is certain to be reduced using constraints. Thus, we derive an expression analogous to (6) for constrained MUSIC:

$$\mathrm{SD}_c \approx \frac{2\sigma^2(m-q)}{N}\sum_{i=1}^{q-q_1}\frac{\lambda_{cs,i}}{(\lambda_{cs,i}-\sigma^2)^2}. \qquad (7)$$

In these expressions, m is the number of sensors, q is the total number of signals, q_1 is the number of known signals, the $\lambda_{s,i}$ are the signal subspace eigenvalues for the original correlation matrix and the $\lambda_{cs,i}$ are the signal subspace eigenvalues for constrained MUSIC. The constraining transformation deflates the signal subspace, so that there are less nonzero signal subspace eigenvalues associated with MUSIC than with constrained MUSIC. To prove $\mathrm{SD}_c \leq \mathrm{SD}$, we first

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substitute a change of variables on the eigenvalues in (6) and (7). Writing $\mu_i = \lambda_i - \sigma^2$ (the μ_i are the eigenvalues of the no noise correlation matrix), and using the Poincaré Separation Theorem (Corollary 4.3.16 of [10]), we have

$$\mu_{s,i} \geq \mu_{cs,i} \geq \mu_{s(i+q_1)}. \tag{8}$$

Thus for every term $\frac{\mu_{cs,i}+\sigma^2}{\mu_{cs,i}^2} = \frac{1}{\mu_{cs,i}} + \frac{\sigma^2}{\mu_{cs,i}^2}$ in (7), there is a corresponding, greater than or equal to term $\frac{\mu_{s,(i+q_1)}+\sigma^2}{\mu_{s,(i+q_1)}^2} = \frac{1}{\mu_{s,(i+q_1)}} + \frac{\sigma^2}{\mu_{s,(i+q_1)}^2}$ in (6). Also, there are q_1 remaining nonzero terms in (6), which gives strict inequality, i.e., this proves $SD_c < SD$.

To verify these results, we compare the distance between the true and estimated signal subspaces obtained with and without the use of constraints in figure 1. (Throughout the rest of this paper, the initials UE refer to unconstrained element space and CE refers to constrained element space.) In this example, the array has 10 equally spaced (half-wavelength) sensors and the signals are from 90 and 92 degrees. (Broadside corresponds to 90 degrees.) The SNR is 20 dB for each signal and 1000 snapshots were used for each of 1000 trials at each point on the estimated variance curves. The intersignal coherence phase is fixed at $\phi = \angle(\mathbf{a}_1^H \mathbf{a}_2)$. This choice of coherence phase will be explained in a later section. The coherence magnitude is allowed to vary between ± 0.9 — negative coherence magnitude actually represents a coherence phase shift of 180 degrees. The key observations from this figure are 1) the estimated and predicted subspace distances are decreased by the use of constraints and 2) the estimated subspace distances closely track their predicted values, with and without the use of constraints. For the negative coherence magnitudes, the subspace distances are actually decreased compared to the case with no inter-signal coherence. For comparison, we have plotted the estimated and predicted DOA variances for UE and CE-MUSIC in figure 2. The situation is identical to that in figure 1, except that DOA variance is plotted instead of subspace distance.

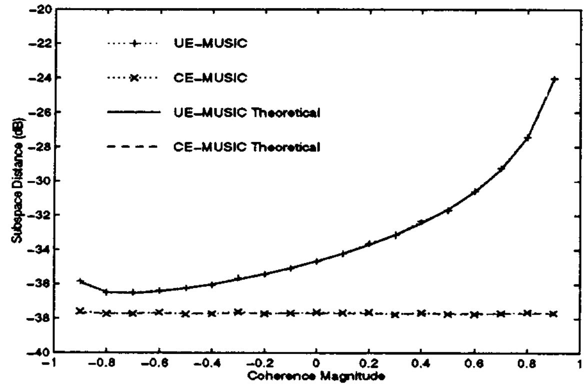

Figure 1: *Distance between the actual and estimated signal subspaces.*

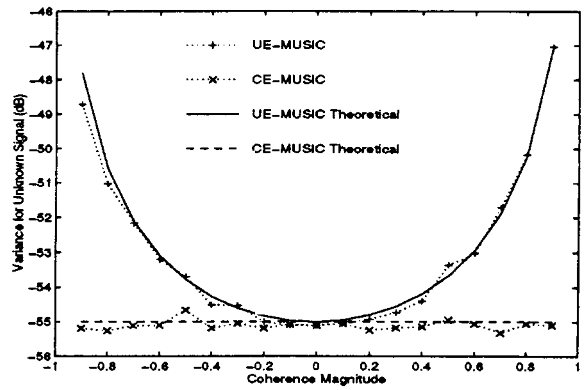

Figure 2: *DOA variance.*

3. COMPARING CONSTRAINED AND UNCONSTRAINED MUSIC

Using constraints improves the direction estimates in most cases, sometimes dramatically, but there are (theoretical) instances where the constraint can make things slightly worse. However, as demonstrated in this section, cases where unconstrained MUSIC is predicted to have lower variance than constrained MUSIC generally correspond to situations where unconstrained MUSIC is in breakdown, but constrained MUSIC is not.

The DOA variance for the various forms of MUSIC can always be put in the following form:

$$\text{var}(\widehat{\theta}_i) = \frac{\sigma^2[\mathbf{G}]_{ii}}{2N[\mathbf{H}]_{ii}} \tag{9}$$

where the matrix $\mathbf{G}$ has terms related to the signal subspace for the relevant correlation matrix and the matrix $\mathbf{H}$ is related to the noise subspace. As the denominator of (9) is the same for either UE-MUSIC or CE-MUSIC, we need only specify $\mathbf{G}$ for comparison purposes. For details related to asymptotic variance of MUSIC and its variations, see [11, 6, 5, 8]. For this discussion, assume that $\mathbf{R}$ is described by

$$\mathbf{R} = \sigma^2\mathbf{I} + [\mathbf{A}_1 \quad \mathbf{A}_2] \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^H & \mathbf{P}_{22} \end{bmatrix} [\mathbf{A}_1 \quad \mathbf{A}_2]^H \tag{10}$$

where $\mathbf{A}_1$ corresponds to the known signals and $\mathbf{A}_2$ corresponds to the unknown signals. Then,

$$\mathbf{G}_{UE} = \mathbf{P}^{-1} + \sigma^2\mathbf{P}^{-1}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{P}^{-1} \tag{11}$$

and

$$\mathbf{G}_{CE} = \mathbf{P}_{22}^{-1} + \sigma^2\mathbf{P}_{22}^{-1}(\mathbf{A}_2^H\mathbf{Q}_{CE}\mathbf{Q}_{CE}^H\mathbf{A}_2)^{-1}\mathbf{P}_{22}^{-1}. \tag{12}$$

CASE 1: If the known signals are highly correlated with the unknown signals, then constrained MUSIC always performs better than MUSIC. If we consider a pair of signals, one known and one unknown, and then allow the correlation between them to vary, as they become completely correlated, $\mathbf{P}$ becomes singular. Thus, the variance of MUSIC approaches infinity in these cases, while the variance of constrained MUSIC does not depend on the coherence, because the constraint effectively removes the coherence when it removes the known signal.

CASE 2: If the SNR is high ($\mathbf{P}_{ii} >> \sigma^2$, for all i) or if $m >> 1$, then the first term in (11) and (12) dominates the second term in (11) and (12). In this case, to compare constrained MUSIC to unconstrained MUSIC, we need only compare $\mathbf{P}_{22}^{-1}$ to the lower right corner of $\mathbf{P}^{-1}$. Using the block matrix inversion relation [12, pg. 23], we have

$$\left(\mathbf{P}_{22} - \mathbf{P}_{12}^H \mathbf{P}_{11}^{-1} \mathbf{P}_{12}\right)^{-1} \tag{13}$$

for the lower right corner of $\mathbf{P}^{-1}$ and we wish to show

$$\mathbf{P}_{22}^{-1} \leq \left(\mathbf{P}_{22} - \mathbf{P}_{12}^H \mathbf{P}_{11}^{-1} \mathbf{P}_{12}\right)^{-1} \tag{14}$$

by which we mean that $\left(\mathbf{P}_{22} - \mathbf{P}_{12}^H \mathbf{P}_{11}^{-1} \mathbf{P}_{12}\right)^{-1} - \mathbf{P}_{22}^{-1}$ is positive semi-definite. Since both $\mathbf{P}_{22}$ and $\mathbf{P}_{22} - \mathbf{P}_{12}^H \mathbf{P}_{11}^{-1} \mathbf{P}_{12}$ are positive definite, we have that (14) is true by Corollary 7.7.4 of [10] and hence the diagonal elements of the lower right corner of $\mathbf{P}^{-1}$ are greater than those of $\mathbf{P}_{22}^{-1}$. This proves that for large N, the variance of constrained MUSIC is always less than that of unconstrained MUSIC for high SNR or large m situations.

CASE 3: If the known signals are uncorrelated with the unknown signals ($\mathbf{P}_{12} = \mathbf{0}$), then the performances of MUSIC and constrained MUSIC for the DOA's unknown to constrained MUSIC are identical (for large N), i.e.,

$$\mathbf{P}_{12} = \mathbf{0} \Rightarrow \text{var}_{UE}(\hat{\theta}_i) = \text{var}_{CE}(\hat{\theta}_i). \tag{15}$$

To see why, first note that $\mathbf{G}_{UE}$ is $q \times q$ whereas $\mathbf{G}_{CE}$ is $(q - q_1) \times (q - q_1)$, thus we must extract the lower right portion of $\mathbf{G}_{UE}$ (corresponding to the unknown signals) in order to compare. If $\mathbf{P}_{12} = \mathbf{0}$,

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{P}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{22}^{-1} \end{bmatrix} \tag{16}$$

so that we need only compare the lower right $q - q_1 \times q - q_1$ block of $(\mathbf{A}^H\mathbf{A})^{-1}$ to $(\mathbf{A}_2^H \mathbf{Q}_{c2} \mathbf{Q}_{c2}^H \mathbf{A}_2)^{-1}$ (see (11) and (12)). Writing

$$\mathbf{A}^H\mathbf{A} = [\mathbf{A}_1 \quad \mathbf{A}_2]^H [\mathbf{A}_1 \quad \mathbf{A}_2] = \begin{bmatrix} \mathbf{A}_1^H\mathbf{A}_1 & \mathbf{A}_1^H\mathbf{A}_2 \\ \mathbf{A}_2^H\mathbf{A}_1 & \mathbf{A}_2^H\mathbf{A}_2 \end{bmatrix}, \tag{17}$$

we have from [12, pg. 23], that

$$\left[\mathbf{A}^H\mathbf{A}\right]^{-1} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^H & \mathbf{B}_{22} \end{bmatrix} \tag{18}$$

where $\mathbf{B}_{22}$ is given by

$$\mathbf{B}_{22} = \left(\mathbf{A}_2^H \left[\mathbf{I} - \mathbf{A}_1(\mathbf{A}_1^H\mathbf{A}_1)^{-1}\mathbf{A}_1^H\right] \mathbf{A}_2\right)^{-1} \tag{19}$$

The term in the brackets, $\mathbf{I} - \mathbf{A}_1(\mathbf{A}_1^H\mathbf{A}_1)^{-1}\mathbf{A}_1^H$, is a projection operator whose span is the orthogonal complement of $\mathbf{A}_1$, which is exactly how $\mathbf{Q}_{CE}\mathbf{Q}_{CE}^H$ is defined (see [5]). So, $\mathbf{B}_{22} = \left(\mathbf{A}_2^H \mathbf{Q}_{CE} \mathbf{Q}_{CE}^H \mathbf{A}_2\right)^{-1}$ (see (12)). This proves that, for large N, the variance of constrained MUSIC is the same as that of MUSIC if the unknown signals are uncorrelated with the known signals.

A Comparison of UE and CE-MUSIC for a Two Signal Scenario. In this case, $\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2]$ where $\mathbf{a}_1$ is a known signal and $\mathbf{a}_2$ is an unknown signal. Also,

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12}^* & p_{22} \end{bmatrix}. \tag{20}$$

We assume that $\mathbf{a}_i^H\mathbf{a}_i = m$, the number of sensors. If we estimate the unknown signal direction using UE-MUSIC and define $\gamma = \mathbf{a}_2^H\mathbf{a}_1$, we have

$$\begin{aligned} \mathbf{G}_{UE}(2,2) &= \frac{p_{11}}{p_{11}p_{22} - |p_{12}|^2} \\ &+ \frac{\sigma^2\left(p_{11}^2 m + p_{11}\left(p_{12}\gamma^* + p_{12}^*\gamma\right) + |p_{12}|^2 m\right)}{\left(p_{11}p_{22} - |p_{12}|^2\right)^2 \left(m^2 - |\gamma|^2\right)} \end{aligned} \tag{21}$$

and using CE-MUSIC with the unknown signal,

$$\mathbf{G}_{CE} = \frac{1}{p_{22}} + \frac{\sigma^2 m}{p_{22}^2\left(m^2 - |\gamma|^2\right)}. \tag{22}$$

To see the effect of the inter-signal correlation, consider p_{12}. It is related to the intersignal coherence or correlation coefficient, c:

$$|c| = \frac{|p_{12}|}{\sqrt{p_{11}p_{22}}} \quad \text{and} \quad \phi = \angle c = \angle p_{12} \tag{23}$$

where ϕ is the coherence phase. Now consider the following term from the second term in the numerator of (21):

$$(p_{12}\gamma^* + p_{12}^*\gamma) = 2\text{Re}(p_{12}\gamma^*) = 2\,|p_{12}|\,(\gamma_R\cos\phi + \gamma_I\sin\phi). \tag{24}$$

where γ_R, γ_I represent the real and imaginary parts of γ. The denominator in (21) does not depend on the coherence phase, thus we can look for minima and maxima of the variance with respect to the coherence phase by looking at the derivative of the numerator only:

$$\frac{\partial \text{var}_{UE}}{\partial \phi} = 0 \quad \Rightarrow \quad -\gamma_R\sin\phi + \gamma_I\cos\phi = 0 \tag{25}$$

or

$$\tan\phi = \frac{\gamma_I}{\gamma_R} \tag{26}$$

so that $\phi = \angle\gamma$ or $\phi = \angle\gamma + \pi$. It is straightforward to verify via a second derivative that $\phi = \angle\gamma$ corresponds to a maximum and $\phi = \angle\gamma + \pi$ corresponds to a minimum. For a fixed magnitude, this coherence phase yields the minimum/maximum variance for unconstrained MUSIC. We can use the above result to search for cases where the predicted variance of unconstrained MUSIC is actually lower than that of constrained MUSIC. In many of the cases we have considered, the minimum variance for unconstrained MUSIC was achieved using a nonzero coherence magnitude.

In figure 3, we plot the predicted variance for both unconstrained and constrained MUSIC over different coherence phases, coherence magnitudes, and different SNR's. The conditions are identical to those in figure 2 except that the SNR and coherence phase are allowed to vary. At each SNR, a family of curves are drawn where each curve corresponds to a different coherence phase. For each plot, a minimum in the variance of unconstrained MUSIC is achieved for a negative coherence magnitude at the coherence phase $\angle\gamma$. Figure 3 seems to indicate that there are some coherence values at which the variance of MUSIC is less than that of constrained MUSIC. To study this issue, we ran simulations comparing UE-MUSIC to CE-MUSIC over a range of SNR's. At each SNR, we used the coherence phase and magnitude which minimized the theoretical variance of UE-MUSIC (obtained from the plots in figure 3) and compared the variance of UE-MUSIC with this best case coherence to that of CE-MUSIC. The potential advantage of MUSIC over constrained MUSIC occurs only at low SNR, and by the time SNR is low enough for the potential advantage to be visible, MUSIC has broken down and the asymptotic variance expressions are irrelevant. See figure 4.

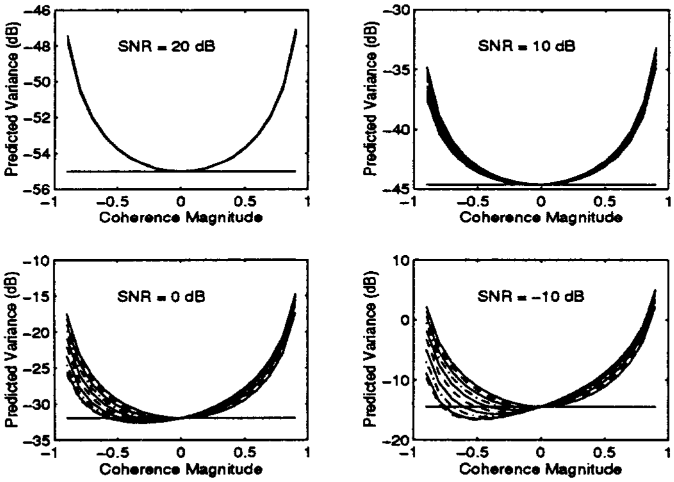

Figure 3: *DOA variance. For 4 different SNR's, we have plotted predicted variance for a range of coherence phases. The most interesting coherence phase corresponds to* $\angle\gamma$*; this phase corresponds to the lowest variance (unconstrained MUSIC) curve for negative coherence magnitudes and the highest variance for the positive coherence magnitudes. At each SNR, the constant line towards the bottom of the plot represents the predicted variance of constrained MUSIC, which is independent of the coherence magnitude.*

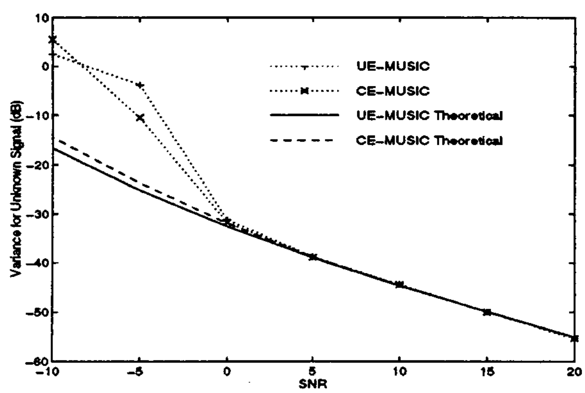

Figure 4: *Variance is plotted in dB as a function of SNR. At each point, the best case (negative) coherence magnitude was chosen for unconstrained MUSIC (from figure 3).*

4. CONCLUSIONS

To compare constrained MUSIC methods to unconstrained methods, we first analyzed the effect of constraints on subspace distance, demonstrating (with analysis and simulations) that constraints always reduce the distance between the estimated and actual subspaces. We were able to prove that constraints always reduce variance under high coherence magnitude, large m, or high SNR conditions. We also showed that cases where unconstrained MUSIC is predicted to outperform constrained MUSIC generally correspond to breakdown conditions for MUSIC. As part of this analysis, we also obtained results related to the effect of coherence on the performance of MUSIC. We derived best and worst case coherence phases, in terms of minimizing/maximizing the variance of unconstrained MUSIC. We have also shown that there are occasions where the variance of unconstrained MUSIC decreases as the signals get more correlated — a somewhat unexpected fact.

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