THEORETICAL COMPARISON FOR BIASES OF MUSIC-LIKE DOA ESTIMATORS

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ABSTRACT

Many MUSIC-like DOA estimators, such as Min-Norm, Beamspace MUSIC, Likelihood MUSIC, FINE and FINES, have been proposed to improve the performance of MUSIC. Since in the difficult estimation situations the large-sample bias of MUSIC may become the dominant estimation error, a comparative study of biases of MUSIC-like estimators in these cases is necessary for their performance evaluation. This paper first identifies the dominant part of the bias of MUSIC for two closely-spaced sources. Then the paper presents a theoretical analysis of a hierarchy of the performances of these MUSIC-like estimators based on their abilities at reducing this major part of the bias and maintaining the asymptotic variance of MUSIC. The theoretical results in the paper explain analytically many previous observations resulting from simulations and numerical computations and may be useful for developing new MUSIC-like algorithms with reduced resolution threshold over that of MUSIC.

I. Introduction

A number of one-dimensional search signal-subspace algorithms such as MUSIC [1], Min-Norm [2], Weighted MUSIC [3], Beamspace MUSIC[4], LMUS (Likelihood MUSIC) [5], Weighted-Norm MUSIC [6], FINE and FINES [7], D_r-estimator [6][8], have been suggested for the estimation of the directions of arrival (DOA). These estimators form a family, which, for convenience, we will call the MUSIC family in this paper.

In the class of one dimensional search DOA estimators, MUSIC is a large-sample solution to the maximum likelihood estimation problem [9]. Thus, it has been shown that all members of the MUSIC family cannot provide a lower asymptotic variance than MUSIC. But many authors have observed that the resolution threshold of MUSIC is bias-driven and some of estimators in the MUSIC family may present smaller biases than MUSIC. Therefore, a comparative study of biases in the MUSIC family is carried out for some members by numerical evaluations of these necessary for their performance evaluation. Unfortunately, general expressions for the biases of these estimators are very complicated. So, the comparisons of bias performance have been expressions or simulations [10],[11].

This paper presents theoretical comparisons of biases in the MUSIC family. These comparisons not only explain analytically many previous observations resulting from simulations, but also provide some further theoretical results.

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II. The Main Part of the Bias of MUSIC

A standard white Gaussian model for a narrowband array with L sensors and d incoherent sources is used in the paper :

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \tag{1}$$

where $\mathbf{A} = [\mathbf{a}(\theta_1^{(0)}), \mathbf{a}(\theta_2^{(0)}), \cdots, \mathbf{a}(\theta_d^{(0)})]$ is the steering matrix, $\theta_1^{(0)}, \theta_2^{(0)}, \cdots, \theta_d^{(0)}$ are d true directions and Rank(A)=d. Suppose

$$\mathbf{e}_{x,1}, \mathbf{e}_{x,2}, \cdots, \mathbf{e}_{x,L}, \lambda_{x,1} > \cdots > \lambda_{x,d} > \lambda_{x,d+1} = \cdots = \lambda_{x,L} = \sigma_n^2 \tag{2}$$

is the eigenstructure of $\mathbf{R}_x = E[\mathbf{x}(t)\mathbf{x}^H(t)] = \mathbf{A}\mathbf{R}_s\mathbf{A} + \sigma_n^2\mathbf{I}$. Let the sample covariance matrix for N snapshots be $\hat{\mathbf{R}}$. Hereafter, the random variables, vectors , matrices and functionals obtained from $\hat{\mathbf{R}}$ will be considered estimates and represented by '^' over the corresponding scalars, vectors, matrices and functionals. The MUSIC estimator is defined as

$$\hat{\theta}^{(2)} = \text{minimizer } \mathbf{v}^H(\theta)[\mathbf{I}_{L\times L} - \hat{\mathbf{E}}_s\hat{\mathbf{E}}_s^H]\mathbf{v}(\theta), \tag{3}$$

where $\mathbf{v}(\theta)$ is the normalized steering vector and $\hat{\mathbf{E}}_s = [\hat{\mathbf{e}}_{x,1}, \cdots, \hat{\mathbf{e}}_{x,d}]$ is the estimator of $\mathbf{E}_s = [\mathbf{e}_{x,1}, \cdots, \mathbf{e}_{x,d}]$. $\text{span}\{\mathbf{e}_{x,1}, \cdots, \mathbf{e}_{x,d}\}$ is called the signal subspace of x(t).

From the expressions of the mean square error and bias of the MUSIC estimator [12],[13], it follows that

$$E(\hat{\theta}^{(2)} - \theta^{(0)})^2 = \frac{1}{N\ddot{D}_2}\sum_{j=1}^{d}\frac{\lambda_{x,j}\sigma_n^2}{(\lambda_{x,j} - \sigma_n^2)^2}\left|\mathbf{e}_{x,j}^H\dot{\mathbf{v}}(\theta)\right|^2 + o(\frac{1}{N}), \tag{4}$$

$$E(\hat{\theta}^{(2)} - \theta^{(0)}) = \frac{1}{N}(L-d-1)P_1^{(2)} + \frac{1}{N}P_2^{(2)} + o(\frac{1}{N}), \tag{5}$$

where

$$P_1^{(2)} = -\frac{1}{\ddot{D}_2}\sum_{j=1}^{d}\frac{\lambda_{x,j}\sigma_n^2}{(\lambda_{x,j} - \sigma_n^2)^2}2\,\text{Re}[\mathbf{e}_{x,j}^H\dot{\mathbf{v}}(\theta)\mathbf{v}^H(\theta)\mathbf{e}_{x,j}],$$
$$P_2^{(2)} = -\frac{\dddot{D}_2}{6\ddot{D}_2^2}\sum_{j=1}^{d}\frac{\lambda_{x,j}\sigma_n^2}{(\lambda_{x,j} - \sigma_n^2)^2}\left|\mathbf{e}_{x,j}^H\dot{\mathbf{v}}(\theta)\right|^2, \tag{6}$$

and

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$$\dot{D}_2 = 2\dot{v}^H(\theta^{(0)})[I - E_s E_s^H]\dot{v}(\theta^{(0)}),$$
$$\ddot{D}_2 = 6\mathrm{Re}\{\ddot{v}^H(\theta^{(0)})[I - E_s E_s^H]\dot{v}(\theta^{(0)})\}. \quad (7)$$

It is well known that in the difficult estimation situations , e.g. large dynamic range, high correlation between sources and very closely-spaced sources, the finite-sample bias of the conventional MUSIC may become the dominant estimation error. Therefore, we focus on the comparisons of bias performances of MUSIC family in the case of two closely-spaced sources.

Let $\Delta = |\theta_1^{(0)} - \theta_2^{(0)}|$. Assuming

$$\left|\dot{v}^H(\theta_1^{(0)})e_2^{(0)}\right| \neq 0 \text{ and } \sum_{k=3}^{L}\left|\dot{v}^H(\theta_1^{(0)})e_k^{(0)}\right|^2 \neq 0, \quad (8)$$

from [14] we can prove that for i=1,2,

$$1)\ \left|v^H(\theta_i^{(0)})e_2\right| = O(\Delta), \quad \left|\dot{v}^H(\theta_i^{(0)})e_2\right| = O_\Delta(1), \quad (9)$$

$$2)\ \sum_{k=d+1}^{L}\left|\dot{v}^H(\theta^{(0)}{}_i)e_d\right|^2 = O(\Delta), \quad (10)$$

where $O_\Delta(1)$ is bounded as $\Delta \to 0$. Based on (9) and (10), we claim that for two closely-spaced sources in (5) $P_1^{(2)} = O(\Delta^{-4})$ and $P_2^{(2)} = O(\Delta^{-3})$. That is, in this case the *main part* of the bias of MUSIC is

$$B_{main}^{(2)} \triangleq \frac{1}{N}(L-3)P_1^{(2)} =$$
$$-\frac{(L-3)}{N\ddot{D}_2}\sum_{j=1}^{2}\frac{\lambda_{x,j}\sigma_n^2}{(\lambda_{x,j}-\sigma_n^2)^2}2\mathrm{Re}[e_{x,j}^H\dot{v}(\theta)v^H(\theta)e_{x,j}] \quad (11)$$

where $\ddot{D}_2$ is given in (7).

III. Reduction of the Bias of MUSIC by Beamspace MUSIC

Assume $y(t)=B^H x(t)$, where B is an $L\times L'$ beamforming matrix, $d<L'\leq L$ and $B^H B = I_{L'\times L'}$. Let R_y be the covariance matrix of $y(t)$ and the eigenvectors of R_y be $e_{y,1}, e_{y,2}, \cdots, e_{y,L'}$. Applying MUSIC algorithm to $y(t)$, we obtain the Beamspace MUSIC estimator of the directions $\theta_1^{(0)}, \theta_2^{(0)}, \cdots, \theta_d^{(0)}$:

$$\hat{\theta}^{(b)} = \text{minimizer } (v'(\theta))^H[I_{L'\times L'} - \hat{E}'_s \hat{E}'^H_s]v'(\theta), \quad (12)$$

where

$$v'(\theta) = \frac{B^H v(\theta)}{\sqrt{v^H(\theta)BB^H v(\theta)}} \quad (13)$$

and $\hat{E}'_s = [\hat{e}_{y,1}, \cdots, \hat{e}_{y,d}]$. It is easily seen that if

$$BB^H A = A, \quad (14)$$

then

$$E(\hat{\theta}^{(b)} - \theta^{(0)})^2 = \frac{1}{N\ddot{D}_b}\sum_{j=1}^{d}\frac{\lambda_{x,j}\sigma_n^2}{(\lambda_{x,j}-\sigma_n^2)^2}\left|e_{x,j}^H v(\theta)\right|^2 + o(\frac{1}{N}) \quad (15)$$

and

$$E(\hat{\theta}^{(b)} - \theta^{(0)}) = \frac{1}{N}(L'-d-1)P_1^{(b)} + \frac{1}{N}P_2^{(b)} + o(\frac{1}{N}), \quad (16)$$

where

$$P_1^{(b)} = -\frac{1}{\ddot{D}_b}\sum_{j=1}^{d}\frac{\lambda_{x,j}\sigma_n^2}{(\lambda_{x,j}-\sigma_n^2)^2}2\mathrm{Re}[e_{x,j}^H\dot{v}(\theta)v^H(\theta)e_{x,j}]$$
$$P_2^{(b)} = -\frac{\dddot{D}_b}{6\ddot{D}_b^2}\sum_{j=1}^{d}\frac{\lambda_{x,j}\sigma_n^2}{(\lambda_{x,j}-\sigma_n^2)^2}\left|e_{x,j}^H v(\theta)\right|^2. \quad (17)$$

In (17),

$$\ddot{D}_b = 2\dot{u}^H(\theta^{(0)})[I - E_s E_s^H]\dot{u}(\theta^{(0)}),$$
$$\dddot{D}_b = 6\mathrm{Re}\{\ddot{u}^H(\theta^{(0)})[I - E_s E_s^H]\dot{u}(\theta^{(0)})\}, \quad (18)$$

where

$$u(\theta) = \frac{BB^H v(\theta)}{\sqrt{v^H(\theta)BB^H v(\theta)}}. \quad (19)$$

In this paper, we always assume that the beamforming matrix satisfies (14).

In this section, we show analytically that under some mild conditions the bias of Beamspace MUSIC will be less than that of conventional MUSIC.

<u>**Theorem 1**</u> Suppose $BB^H[A, \dot{v}(\theta_1^{(0)})] = [A, \dot{v}(\theta_1^{(0)})]$. If

$$P_1^{(2)}P_2^{(2)} > 0 \text{ or } \left|P_1^{(2)}\right| > \frac{2}{L-d-1}\left|P_2^{(2)}\right|, \quad (20)$$

then regardless of the terms of $o(N^{-1})$, it holds that

$$\left|E(\hat{\theta}_1^{(b)} - \theta_1^{(0)})\right| < \left|E(\hat{\theta}_1^{(2)} - \theta_1^{(0)})\right|. \quad (21)$$

Furthermore, for two closely-spaced sources we have

<u>**Theorem 2**</u> If $BB^H A = A$ and L'=3, then

$$\left|\frac{E(\hat{\theta}^{(b)} - \theta^{(0)})}{E(\hat{\theta}^{(2)} - \theta^{(0)})}\right| = O(\Delta) + \varepsilon_N, \quad (22)$$

where $\varepsilon_N \to 0$ as $N \to \infty$.

<u>**Theorem 3**</u> If $BB^H[A, D]=[A, D]$, where $D = [\frac{d}{d\theta}a(\theta_1^{(0)}), \frac{d}{d\theta}a(\theta_2^{(0)})]$, then

$$\left|\frac{E(\hat{\theta}^{(b)} - \theta^{(0)})}{E(\hat{\theta}^{(2)} - \theta^{(0)})}\right| = \frac{L'-3}{L-3} + O(\Delta) + \varepsilon_N, \quad (23)$$

where $\varepsilon_N \to 0$ as $N \to \infty$.

Obvious, if $BB^H[A, D]=[A, D]$ then

$$E(\hat{\theta}^{(2)} - \theta^{(0)})^2 = E(\hat{\theta}^{(b)} - \theta^{(0)})^2 + o(\frac{1}{N}). \quad (24)$$

Therefore, if we choose B such that L'=3 and $BB^H[A, \dot{v}(\theta_1^{(0)})] = [A, \dot{v}(\theta_1^{(0)})]$, then $BB^H\dot{v}(\theta_2^{(0)}) \approx \dot{v}(\theta_2^{(0)})$ for the two closely-spaced sources. This means that (24) approximately holds for both $\theta_1^{(0)}$ and $\theta_2^{(0)}$. This B should be the best choice for both of reducing the bias and maintaining the asymptotic variance of MUSIC. Simulations demonstrate this conclusion is correct.

In short, Beamspace MUSIC is an approach to reduce the bias of MUSIC and the limitation of this reduction for two closely-spaced sources is the removal of the main part of the bias of MUSIC, $\frac{1}{N}(L-d-1)P_1^{(2)}$.

IV. Elimination of the Main Part of the Bias of MUSIC by Likelihood MUSIC

The Weighted-Norm MUSIC estimator is defined as $\hat{\theta}^{(w)} = \text{minimizer}\, \hat{W}(q)\hat{D}_2(q)$, where $\hat{W}(\theta)$ is the estimate of the positive weighting function $W(\theta)$ and $\hat{D}_2(\theta)$ is the estimated null-spectrum of MUSIC. The relations between the large-sample biases and mean-square errors of Weighted-Norm MUSIC and MUSIC , which have been derived in [15], are as follows.

$$E(\hat{\theta}^{(w)} - \theta^{(0)})^2 = E(\hat{\theta}^{(2)} - \theta^{(0)})^2 + o(\frac{1}{N}), \quad (25)$$

and

$$E\hat{\theta}^{(w)} - \theta^{(0)} = E\hat{\theta}^{(2)} - \theta^{(0)} - \frac{1}{N}\frac{\dot{W}_\theta}{W_\theta}\frac{1}{\ddot{D}_{2\theta}}(L-d-1)\sum_{j=1}^{d}\frac{\lambda_j\sigma_n^2}{(\lambda_j-\sigma_n^2)^2}\left|\mathbf{e}_j^H \mathbf{v}(\theta^{(0)})\right|^2 + o(\frac{1}{N}). \quad (26)$$

Likelihood MUSIC (LMUS) algorithm was first proposed by Sharman and Darrani [5] based on minimization of a log-likelihood function. LMUS is a specific weighted-norm MUSIC with a weighting function

$$W_s(\theta) = \frac{1}{\sum_{k=1}^{d}\frac{\lambda_k\sigma_n^2}{(\lambda_k-\sigma_n^2)^2}\left|\mathbf{v}^H(\theta)\mathbf{e}_k\right|^2}. \quad (27)$$

Denote the LMUS estimator by $\hat{\theta}^{(s)}$. Then, we claim that

Theorem 4. $E(\hat{\theta}^{(s)} - \theta^{(0)})^2 = E(\hat{\theta}^{(2)} - \theta^{(0)})^2 + o(\frac{1}{N})$ and

$$E(\hat{\theta}^{(s)} - \theta^{(0)}) = \frac{1}{N}P_2^{(2)} + o(\frac{1}{N}). \quad (28)$$

This means that the main part of the bias of MUSIC can be removed by LMUS. Using the results in the previous section, for two closely-spaced sources we have

Theorem 5. If B satisfies $\mathbf{BB}^H[\mathbf{A}, \dot{\mathbf{v}}(\theta_1^{(0)})] = [\mathbf{A}, \dot{\mathbf{v}}(\theta_1^{(0)})]$, and L'=3, then $E(\hat{\theta}_1^{(s)} - \theta_1^{(0)})^2 = E(\hat{\theta}_1^{(b)} - \theta_1^{(0)})^2 + o(\frac{1}{N})$ and

$$E(\hat{\theta}_1^{(s)} - \theta_1^{(0)}) = E(\hat{\theta}_1^{(b)} - \theta_1^{(0)}) + o(\frac{1}{N}). \quad (29)$$

This assertion is from large-sample theory. For finite sample cases, we only can claim that the best performance of Beamspace MUSIC should be close to LMUS.

V. Biases of Min-Norm, Weighted MUSIC, FINE and FINES

According to the definitions of Min-Norm [2], FINE and FINES [7], they can be regarded as the specific members of the following subclass of weighted MUSIC estimators based on the cost functional

$$F(\theta) = \mathbf{v}^H(\theta)(\mathbf{E}_n\mathbf{E}_n^H)\mathbf{W}(\mathbf{E}_n\mathbf{E}_n^H)\mathbf{v}(\theta), \quad (30)$$

where $\mathbf{E}_n = [\mathbf{e}_{x,d+1}, \cdots, \mathbf{e}_{x,L}]$, $\mathbf{E}_n\mathbf{E}_n^H = \mathbf{I}_{L\times L} - \mathbf{E}_s\mathbf{E}_s^H$ and $\mathbf{W}$ is an (L-d)×(L-d) projection matrix . That is $\mathbf{W}=\mathbf{W}^H$ and $\mathbf{W}^2=\mathbf{W}$. $\text{span}\{\mathbf{e}_{x,d+1}, \cdots, \mathbf{e}_{x,L}\}$ is called the noise subspace of $\mathbf{x}(t)$. (30) is equivalent to

$$F(\theta) = \mathbf{v}^H(\theta)\mathbf{E}'_n\mathbf{E}'^H_n\mathbf{v}(\theta), \quad (31)$$

where $\mathbf{E}'_n\mathbf{E}'^H_n$ is a projection on to some subspace of the noise subspace. Define $\mathbf{B}=[\mathbf{E}_s,\mathbf{E}_n']$ which is an L×L' matrix and satisfies $\mathbf{BB}^H\mathbf{A}=\mathbf{A}$. L' is equal to the sum of the number of sources and the number of columns of $\mathbf{E}_n'$. Obviously, d<L'≤L. Thus, (31) is equivalent to

$$\begin{aligned} F(\theta) &= (\mathbf{v}^H(\theta)\mathbf{BB}^H\mathbf{v}(\theta))\mathbf{u}^H(\theta)(\mathbf{I}_{L\times L} - \mathbf{E}_s\mathbf{E}_s^H)\mathbf{u}(\theta) \\ &= W(\theta)[(\mathbf{v}'(\theta))^H(\mathbf{I}_{L'\times L'} - \mathbf{E}'_s\mathbf{E}'^H_s)\mathbf{v}'(\theta)], \end{aligned} \quad (32)$$

where $W(\theta) = \mathbf{v}^H(\theta)\mathbf{BB}^H\mathbf{v}(\theta)$. This means that (30) is equivalent to a weighted norm Beamspace MUSIC with weighting function $W(\theta) = \mathbf{v}^H(\theta)\mathbf{BB}^H\mathbf{v}(\theta)$. Furthermore, since $\dot{W}_\theta(\theta^{(0)}) = 0$, applying (26) to the Weighted Norm Beamspace MUSIC, we assert that the bias and mean square error of the estimators generated from the sample form of (30) and $[(\mathbf{v}'(\theta))^H(\mathbf{I}_{L'\times L'} - \mathbf{E}'_s\mathbf{E}'^H_s)\mathbf{v}'(\theta)]$ in (32) are the same. That is, FINE, FINES, Min-Norm and the subclass of the weighted MUSIC with the cost functional defined by (30) are specific members of Beamspace MUSIC. Particularly, all of them satisfy $\mathbf{BB}^H\mathbf{A}=\mathbf{A}$. Therefore, Theorems 1, 2 and 3 can also be used for the comparison of biases of FINE, FINES and Min-Norm with the conventional MUSIC. Thus, we claim that for two closely-spaced sources Min-norm and Fine can be used to remove the main part of the bias of MUSIC. Since for these two algorithms, the corresponding beamformer matrix $\mathbf{B}$ does not satisfy $\mathbf{BB}^H[\mathbf{A}, \dot{\mathbf{v}}(\theta_1^{(0)})] = [\mathbf{A}, \dot{\mathbf{v}}(\theta_1^{(0)})]$, their large-sample mean square errors are larger than MUSIC. For FINES, d+1<L'<L. Thus, in this case its bias is less than MUSIC but greater than FINE.

VI. Simulations and Conclusion

Some simulation examples are also presented in the paper to substantiate the theoretical analyses. For example, Fig. 1 and Fig 2 show the biases and RMS' (Root of Mean Square Error) versus Δ of MUSIC, LMUS , Min-Norm and Beamspace MUSIC with L'=3, d=2 and $\mathbf{BB}^H[\mathbf{A}, \dot{\mathbf{v}}(\theta_1^{(0)})] = [\mathbf{A}, \dot{\mathbf{v}}(\theta_1^{(0)})]$ at $\theta_1^{(0)} = 14°$, where N=1000 and the signal-to-noise ratio (SNR) just reaches the 100% resolution threshold. Fig. 3 presents the resolution probabilities of MUSIC, LMUS , Min-Norm and this Beamspace MUSIC in an array with L=10, N=64, and two equipowered and uncorrelated sources at $\theta_1^{(0)} = 14°$ and $\theta_2^{(0)} = 16°$. SNR varies from 4 dB to 24 dB. all the statistics in these figures are based on 200 independent trials. It is clearly seen that the performance of the best Beamspace MUSIC is close to that of LMUS.

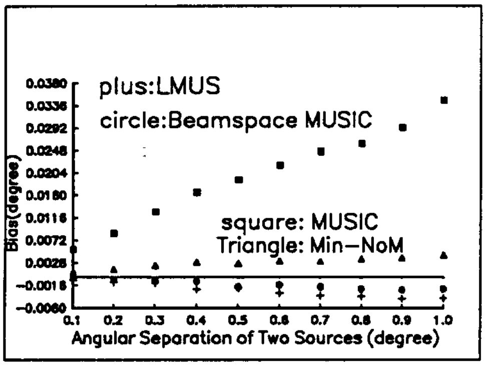

Fig. 1. Comparison of Biases

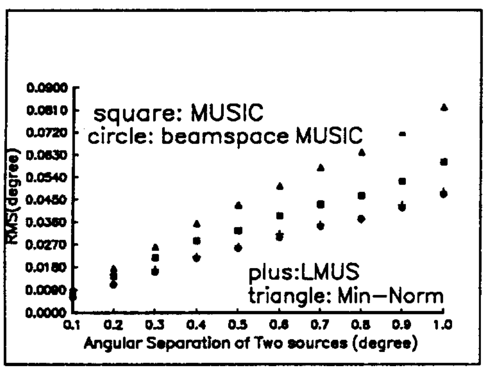

Fig. 2. Comparison of Root of Mean Square Errors

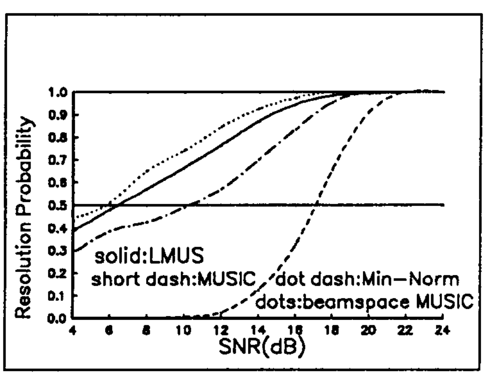

Fig. 3. Comparison of Resolution Probabilities

To summarize, this paper has presented a study of a hierarchy of biases of the MUSIC family such that one can properly evaluate their performance from large-sample biases. This study may also be useful for developing new one-dimensional search methods with reduced resolution threshold over that of MUSIC.

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