ZIV-ZAKAI LOWER BOUNDS IN BEARING ESTIMATION

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ABSTRACT

Bounds on the MSE in estimating the bearing of a planewave signal is of considerable interest in many fields. Of particular importance is the ability of a bound to closely characterize performance in the small error or asymptotic region, and the large error or ambiguity region, and to accurately predict the location of the threshold between the regions. In this paper, the vector Ziv-Zakai bound is applied to the problem of estimating two-dimensional bearing with planar arrays of arbitrary geometry. The bound is calculated for square and circular arrays, and compared with the Weiss-Weinstein bound. The Ziv-Zakai bound is shown to be tighter than the Weiss-Weinstein bound in the threshold and asymptotic regions.

1. INTRODUCTION

Bounds on the MSE in estimating the bearing of a planewave signal are of considerable interest in many fields including radar, sonar, seismic analysis, radio telemetry, medical imaging, and anti-jam communications. Of particular importance in the bearing estimation problem, as well as in the related problem of time delay estimation, is the ability of a bound to closely characterize performance in the small error or asymptotic region, and the large error or ambiguity region, and to accurately predict the location of the threshold between the regions.

Much of the work on performance bounds for bearing estimation has focused on the Cramer Rao bound and Barankin bound, which have been derived for many signal models. However, it is well known that the Cramer-Rao bound is useful only in the asymptotic region (i.e. high SNR and/or long observation times) and is applicable only to unbiased estimators. The Barankin bound can be difficult to use as it requires optimization over a large number of free parameters, and is also applicable only to unbiased estimators. In the bearing estimation problem, the parameter space is limited to a finite interval and no estimators can be constructed which are unbiased over the entire interval, therefore these bounds may not adequately characterize performance of real estimators.

The Ziv-Zakai bound (ZZB) and Weiss-Weinstein bound (WWB) are free from bias assumptions and have been successfully applied to time delay and bearing estimation problems [3]-[6]. They are some of the tightest bounds available for all regions of operation. Previously, the application of the ZZB was restricted to the important but limited class of problems involving a single uniformly distributed parameter, such as estimating the bearing of one source with a linear array. For these problems, the ZZB is tighter in the threshold and asymptotic regions than the WWB [5]. The WWB has also been used to analyze performance of planar arrays in estimating the two-dimensional bearing of a source [6]. The ZZB has recently been extended to handle vectors of parameters with arbitrary prior distributions [1]-[2], and can now be applied to this problem. In this paper, the Ziv-Zakai bound on the MSE in estimating bearing with planar arrays of arbitrary geometry is developed. The bound is calculated for a square and circular arrays, and compared with the WWB. As in the scalar problems, the ZZB is tighter in the regions of interest.

2. VECTOR ZIV-ZAKAI BOUND

Consider estimation of a K-dimensional vector random parameter θ with pdf $p(\theta)$, based upon noisy observations $\mathbf{x}$. Let $p(\mathbf{x}|\theta)$ denote the conditional pdf of $\mathbf{x}$ given θ. For any estimator, $\hat{\theta}(\mathbf{x})$, the estimation error is $\epsilon = \hat{\theta}(\mathbf{x}) - \theta$, and the error correlation matrix is defined as

$$\mathbf{R}_\epsilon = E\left\{\epsilon\epsilon^T\right\}. \tag{1}$$

We are interested in lower bounding $\mathbf{a}^T\mathbf{R}_\epsilon\mathbf{a}$, for any K-dimensional vector $\mathbf{a}$. The ZZB has the form:

$$\mathbf{a}^T\mathbf{R}_\epsilon\mathbf{a} \geq \int_0^\infty \frac{\Delta}{2}\cdot V\left\{\max_{\delta\,:\,\mathbf{a}^T\delta=\Delta} f(\delta)\right\} d\Delta, \tag{2}$$

where

$$f(\delta) = \int \left(p(\theta)+p(\theta+\delta)\right) P_{\min}(\theta,\theta+\delta)\,d\theta \tag{3}$$

$$\geq \int 2\min\left(p(\theta),p(\theta+\delta)\right) P_{\min}^{el}(\theta,\theta+\delta)\,d\theta. \tag{4}$$

$P_{\min}(\theta,\theta+\delta)$ is the minimum probability of error in the binary detection problem:

$$\begin{aligned} H_0 : \theta; \quad & \Pr(H_0) = \frac{p(\theta)}{p(\theta)+p(\theta+\delta)}; \\ H_1 : \theta+\delta; \quad & \Pr(H_1) = 1-\Pr(H_0), \end{aligned} \tag{5}$$

$P_{\min}^{el}(\theta,\theta+\delta)$ is the minimum probability of error in the same binary detection problem but with equally likely hypotheses, and $V\{\cdot\}$ is the valley-filling function. The bound may be used with either (3) or (4). When the prior density of the parameters is uniform, the two forms are equal.

In applying the bound, one has to specify $\mathbf{a}$. The choice for $\mathbf{a}$ is dictated by the particular parameter or linear combination of parameters being investigated. When a bound on the MSE of the i^{th} component of θ is desired, then $\mathbf{a}$ is the unit vector with a one in the i^{th} position.

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To obtain the tightest bound, we must maximize over the vector δ, subject to the constraint $\mathbf{a}^T\delta = \Delta$. This vector determines the position of the second hypothesis relative to the first in the detection problem (5). In order to satisfy the constraint, δ must be composed of a fixed component along the a-axis, $\frac{\Delta}{\|\mathbf{a}\|^2}\mathbf{a}$, and an arbitrary component orthogonal to a. Thus δ has the form

$$\delta = \frac{\Delta}{\|\mathbf{a}\|^2}\mathbf{a} + \mathbf{b}, \tag{6}$$

where b is an arbitrary vector orthogonal to a, i.e., $\mathbf{a}^T\mathbf{b} = 0$, and we have $K-1$ degrees of freedom in choosing δ via the vector b. In the maximization, δ should be chosen so that the two hypotheses are as indistinguishable as possible by the optimum detector, and therefore produce the largest probability of error. Choosing $\mathbf{b} = \mathbf{0}$ results in the hypotheses being separated by the smallest Euclidean distance. This is often a good choice, but hypotheses separated by the smallest Euclidean distance do not necessarily have the largest probability of error, and maximizing over δ can improve the bound.

3. ZZB FOR BEARING ESTIMATION

Consider the bearing estimation problem in which a narrowband planewave signal impinges on a planar array of M sensors as illustrated in Figure 1. The i^{th} sensor is located at

$$\mathbf{d}_i = \begin{bmatrix} d_{x,i} \\ d_{y,i} \end{bmatrix} \tag{7}$$

in the $x-y$ plane. The bearing has two components which can be expressed in either angular or Cartesian (wavenumber) coordinates :

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta \end{bmatrix}. \tag{8}$$

The observed waveform at i^{th} sensor consists of a delayed version of the source signal and additive noise:

$$x_i(t) = s(t-\tau_i) + n_i(t) \qquad -\frac{T}{2} \le t \le \frac{T}{2} \tag{9}$$

where

$$\tau_i = \frac{\mathbf{u}^T\mathbf{d}_i}{c} \tag{10}$$

and c is the speed of propagation. We assume that the source and noise waveforms are sample functions of independent, zero-mean, Gaussian random processes with known spectra. The source's spectrum is

$$P(\omega) = \begin{cases} P & |\omega-\omega_0| \le \frac{W}{2} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

where $\frac{W}{\omega_0} \ll 1$, and the noise is white, $N(\omega) = \frac{N_0}{2}$. We wish to estimate the two-dimensional wavenumber u.

To evaluate the vector bound (2), we need an expression for $P_{\min}(\mathbf{u},\mathbf{u}+\delta)$ or $P_{\min}^{el}(\mathbf{u},\mathbf{u}+\delta)$. In this problem, neither $P_{\min}(\mathbf{u},\mathbf{u}+\delta)$ nor $P_{\min}^{el}(\mathbf{u},\mathbf{u}+\delta)$ can be written in

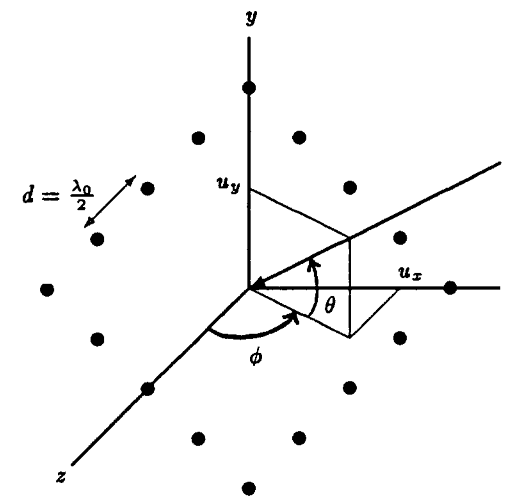

Figure 1: Geometry of the single source bearing estimation problem with 16-element square array.

closed form, however a tight lower bound for *equally likely* hypotheses is given by [7],[4]:

$$P_{\min}^{el}(\mathbf{u},\mathbf{u}+\delta) \ge P_b^{el}(\delta)$$
$$\equiv \exp\left\{\mu\left(\tfrac{1}{2};\delta\right) + \frac{1}{8}\ddot{\mu}\left(\tfrac{1}{2};\delta\right)\right\}\Phi\left(-\frac{1}{2}\sqrt{\ddot{\mu}\left(\tfrac{1}{2};\delta\right)}\right) \tag{12}$$

where $\mu(s;\delta)$ is the log of the semi-invariant moment generating function

$$\mu(s;\delta) = \ln\int p(\mathbf{x}|\mathbf{u}+\delta)^s p(\mathbf{x}|\mathbf{u})^{1-s}d\mathbf{x}, \tag{13}$$

and $\ddot{\mu}(s;\delta)$ is the second derivative of $\mu(s;\delta)$ with respect to s. Although bounds on $P_{\min}(\mathbf{u},\mathbf{u}+\delta)$ are available, the best MSE bound for this problem was obtained using (12) in (4). Note that the probability of error bound is only a function of δ and not a function of u. This simplifies the computation of the bound.

For the bearing estimation problem under consideration, $\mu(s;\delta)$ has the simple form (see e.g. [3]-[4])

$$\mu(s;\delta) = -\frac{WT}{2\pi}\ln\left[1+s(1-s)\beta(\delta)\right] \tag{14}$$

where

$$\beta(\delta) \equiv \gamma\left(1-|\rho(\delta)|^2\right) \tag{15}$$

$$\gamma \equiv \frac{\left(\frac{2MP}{N_0}\right)^2}{1+\left(\frac{2MP}{N_0}\right)} \tag{16}$$

$$\rho(\delta) \equiv \frac{1}{M}\mathbf{E}(\mathbf{u}+\delta)^\dagger\mathbf{E}(\mathbf{u}), \tag{17}$$

$$\mathbf{E}(\mathbf{u}) = \begin{bmatrix} e^{-j\frac{\omega_0}{c}\mathbf{u}^T\mathbf{d}_1} \\ \vdots \\ e^{-j\frac{\omega_0}{c}\mathbf{u}^T\mathbf{d}_M} \end{bmatrix}. \tag{18}$$

When $s = \frac{1}{2}$, we get

$$\mu(\tfrac{1}{2};\delta) = -\frac{WT}{2\pi}\ln\left(1+\frac{1}{4}\beta(\delta)\right) \quad (19)$$

$$\ddot{\mu}(\tfrac{1}{2};\delta) = \frac{WT}{2\pi}\frac{2\beta(\delta)}{1+\frac{1}{4}\beta(\delta)}. \quad (20)$$

The final bound has the form

$$\mathbf{a}^T\mathbf{R}_\epsilon\mathbf{a} \geq \int_0^2 \Delta\cdot\mathcal{V}\left\{\max_{\delta\,:\,\mathbf{a}^T\delta=\Delta} P_b^{el}(\delta)A(\delta)\right\}d\Delta. \quad (21)$$

where

$$A(\delta) = \int \min(p(\mathbf{u}), p(\mathbf{u}+\delta))d\mathbf{u}. \quad (22)$$

Evaluation of $P_b^{el}(\delta)$ depends on the geometry of the array and $A(\delta)$ depends on the a priori distribution of u. If the vector wavenumber is uniformly distributed on the unit disc:

$$p(\mathbf{u}) = \begin{cases} \frac{1}{\pi} & \sqrt{u_x^2+u_y^2} \leq 1 \\ 0 & \text{otherwise,} \end{cases} \quad (23)$$

then $A(\delta)$ is π^{-1} times the area of intersection of two unit circles centered at the origin and δ:

$$A(\delta) = 2\arccos\left(\frac{\|\delta\|}{2}\right) - \sin\left(2\arccos\left(\frac{\|\delta\|}{2}\right)\right). \quad (24)$$

In order to evaluate the ZZB, the expressions (12)-(20) and (24) must be substituted into (21) and computed numerically. The WWB for this problem can be found in [6].

4. PERFORMANCE OF SQUARE AND CIRCULAR ARRAYS

The bound derived in the previous section can be applied to arrays of any geometry. In this section we investigate the performance of square and circular arrays. Both arrays have $M = 16$ elements with sensors evenly spaced $\frac{\lambda_0}{2}$ apart. The square array is shown in Figure 1. Both of the arrays are symmetric in the x and y directions, therefore the MSE in estimating u_x and u_y will be the same. To evaluate the MSE in estimating u_x, we choose $\mathbf{a} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

The ZZB and WWB, normalized with respect to the prior variance of u_x, are plotted vs. SNR in Figure 2 for the square array. Several versions of the ZZB are plotted to illustrate the impact of valley-filling and maximizing over δ. All of the bounds approach the prior variance for small SNR, and the WWB is tighter than the ZZB in this region. In the threshold and high SNR regions, the use of valley-filling alone and maximization over δ without valley-filling produces bounds which are slightly better than the WWB. Combining maximization with valley-filling yields a bound which is significantly better than the other bounds in this region. To understand this, it is necessary to look at the function $f(\delta) = P_b^{el}(\delta)A(\delta)$, which is plotted in Figure 3.

This function is the probability of error for two hypotheses separated by δ, weighted by the function $A(\delta)$, which is decreasing in $\|\delta\|$. Because of the geometry of the array, the probability of error is high not only for small values of δ, but for values of δ which lie on the axes $\delta_2 = \delta_1$ and $\delta_2 = -\delta_1$.

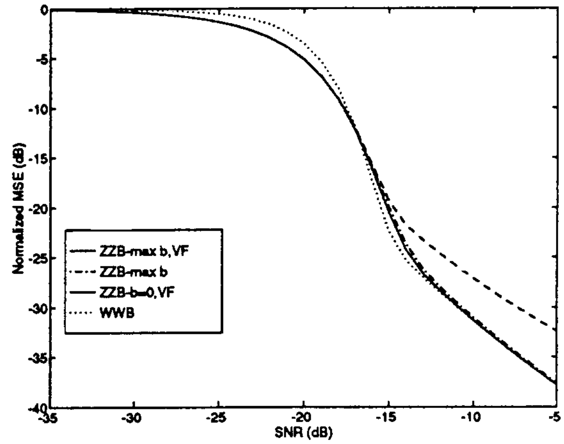

Figure 2: Comparison of the ZZB and WWB for the square array for $\frac{WT}{2\pi} = 100$.

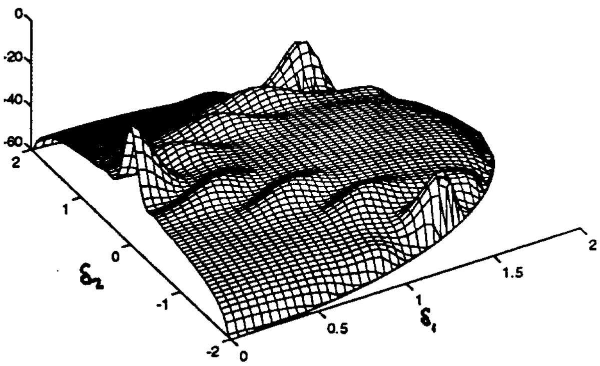

Figure 3: The function $f(\delta)$ for the square array for SNR=-14 dB and $\frac{WT}{2\pi} = 100$.

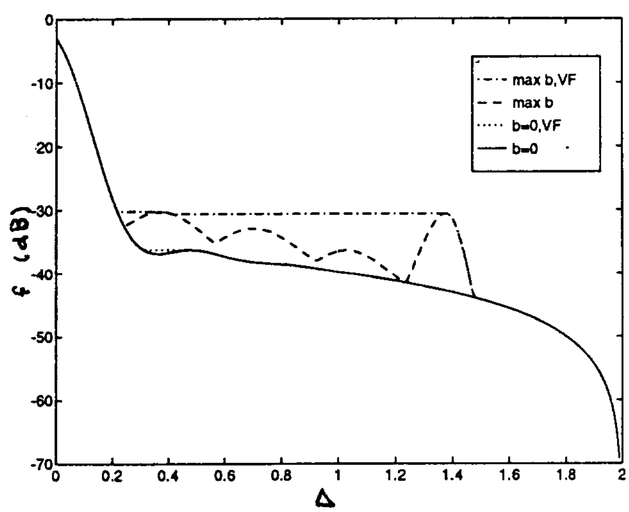

Figure 4: Impact of maximization and valley-filling for square array for SNR=-14 dB and $\frac{WT}{2\pi} = 100$.

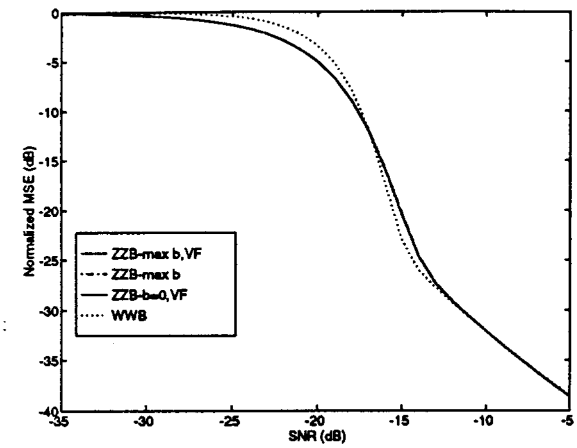

Figure 5: Comparison of the ZZB and WWB for the circular array for $\frac{WT}{2\pi} = 100$.

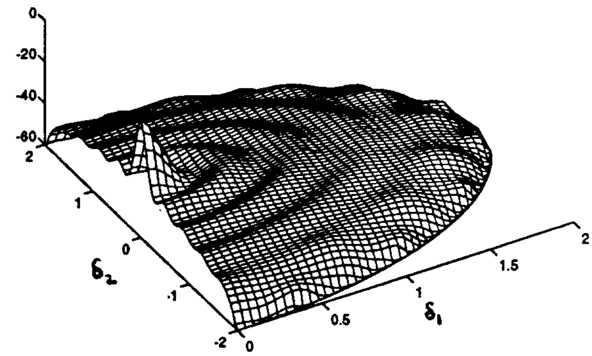

Figure 6: The function $f(\delta)$ for the circular array for SNR=-14 dB and $\frac{WT}{2\pi} = 100$.

The points along these axes are points of ambiguity for the estimation problem as well as the detection problem, and estimation errors will tend to occur more often in these directions. The maximization and valley-filling procedures capture the effects of the ambiguities in the overall MSE.

Optimization of the bound requires maximization of $f(\delta)$ over δ for each value of Δ, subject to the constraint $\mathbf{a}^T\delta = \Delta$, followed by valley-filling. In this problem that means δ must have the form $\delta = \left[\ \Delta \quad b(\Delta)\ \right]^T$, where $b(\Delta)$ can be chosen arbitrarily. When $b = 0$, $f(\delta)$ is evaluated along the line $\delta_2 = 0$, but the bound can be improved by choosing $b(\Delta)$ so that δ lies on one of the diagonal axes. In Figure 4, the results of maximizing $f(\delta)$ over δ, valley-filling $f(\delta)$ with $b(\Delta) = 0$, and valley-filling the maximized function are plotted. The maximized and valley-filled function is significantly larger than the other functions, and when it is integrated, a tighter bound is produced.

In Figure 5, the normalized ZZB and WWB are plotted vs. SNR for the circular array. Once again, both the ZZB and WWB approach the prior variance for small SNR, with the WWB being tighter than the ZZB in this region. The ZZB is tighter in the threshold region, and both bounds converge for high SNR. For this array, valley-filling and maximizing over δ do not improve the bound significantly. The function $f(\delta) = P_b^{el}(\delta)A(\delta)$, plotted in Figure 6, is smoother for the circular array and there are no significant ambiguities as with the square array.

5. CONCLUSION

The vector Ziv-Zakai bound was applied to the problem of estimating two-dimensional bearing with planar arrays of arbitrary geometry. The bound was evaluated for square and circular arrays, and compared with the Weiss-Weinstein bound. The square and circular arrays have nearly the same performance in the low SNR and threshold regions, but the circular array performs better than the square array for large SNR. The square array does not possess as much symmetry as the circular array, resulting in larger estimation errors for some directions of arrival. The ZZB reflects the presence of these large errors for high SNR, while the WWB does not.

6. REFERENCES

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