THE USE OF THE BARANKIN BOUND FOR DETERMINING THE THRESHOLD SNR IN ESTIMATING THE BEARING OF A SOURCE IN THE PRESENCE OF ANOTHER

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ABSTRACT

In this paper we report results of a research in which we studied the problem of determining the threshold signal to noise ratio (SNR) between large and small errors estimation of the direction of arrival (DOA) of a radiating, far-field source in the presence of another. Using the Barankin lower bound (BB) we examine the conditions under which achievable mean square error (mse) performance of any unbiased DOA estimator deviates substantially from the Carmer-Rao lower bound (CRB). We present expressions for the threshold SNR as a function of the source-array geometry and the sources SNR where one and two sources, of known/unknown spectral parameters and DOAs, are present.

1. INTRODUCTION

Passive localization of sources is an active field of research in the last 20 years. Many different algorithms have been proposed for estimating the direction of arrival (DOA) of the sources. Their performance is usually compared to the Cramer-Rao lower bound (CRB), which bounds the variance of any unbiased estimator of the DOAs. This bound has also been studied to gain insight on the inherent limitations of this important estimation problem. The use of the CRB is usually justified by appealing to an asymptotic theorem which asserts that this bound can be closely approached by "sufficiently large" SNR and/or observation time. Unfortunately, in many practical problems assuming asymptotic conditions is unreasonable. Practically, the attainable mse of any unbiased estimate of the DOA deviates radically from the CRB as SNR reduces beyond a critical value, exhibiting a threshold phenomenon. The fast performance degradation below the threshold makes the value of the threshold SNR a critical parameter in system design. Thus, improved bounds are used to predict the threshold SNR value and to analyze reachable DOA estimation performance below threshold in the ambiguity-dominated regime, where some mistaken models can not be distinguished from the true one.

Improved lower bounds on the mse of location-related parameter estimation of a single source for establishing the threshold SNR have been studied in several works (e.g. [1]-[4]). However, with more than one source, only the active case were studied. Ambiguity phenomenon and other "resolution thresholds" in the passive multi-source case were studied by other means (e.g. [5]-[7]). To the best of our knowledge, our study is the first in which improved non-Bayesian bounds (i.e., the Barankin Bound) are used to predict the threshold SNR in a multi-source, passive DOA estimation. (The active case is studied in [8]). We present expressions for the threshold SNR of the estimation error of the DOA of a source in the presence of spatially uncorrelated noise and in the presence of more sources of known/unknown DOAs. In particular, we study the effect of the prior knowledge of nuisance parameters (as the sources spectra) on the threshold SNR.

2. PROBLEM FORMULATION AND BACKGROUND

2.1. Problem formulation

We assume an array of M omnidirectional sensors in an arbitrary geometry which receives zero-mean, Gaussian, narrowband signals radiated from $N = 2$ far-field point sources[1]. The additive noise in each of the sensors is assumed zero-mean, Gaussian, stationary process, statistically independent of the noise at the other sensors.

The signal received at the m-th sensor is described by

$$x_m(t) = \sum_{n=1}^{N} s_n(t - \tau_m(\theta_n)) + n_m(t) \quad \text{for} \quad m = 1, \ldots, M \tag{1}$$

where $s_n(\cdot)$ is, in the passive case, an unknown signal radiated by the n-th source, θ_n is the DOA of the n-th source, $n_m(\cdot)$ is the additive noise at the m-th sensor, and $\tau_m(\theta_n)$ is the propagation delay associated with the n-th source and the m-th sensor. For far-field sources, it is given by: $\tau_m(\theta_n) = \frac{1}{c}(x_m \sin(\theta_n) + y_m \cos(\theta_n))$ where c is the propagation velocity and (x_m, y_m) are the Cartesian coordinates on the m-th sensor. In a matrix form, we may write (1) as:

$$\underline{X}_j = A_j \underline{S}_j + \underline{N}_j \quad \text{for} \quad j = 1, \ldots, L \tag{2}$$

where

$$\underline{X}_j = [X_1(f_j), X_2(f_j), \ldots, X_M(f_j)]^T ,$$
$$\underline{S}_j = [S_1(f_j), S_2(f_j), \ldots, S_N(f_j)]^T ,$$

[1] In [9] we present a simple generalization of the results to more than two uncorrelated sources.

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$$\underline{N}_j = [N_1(f_j), N_2(f_j), \ldots, N_M(f_j)]^T ,$$
$$A_j = [\underline{v}_{\theta_1}(f_j), \underline{v}_{\theta_2}(f_j), \ldots, \underline{v}_{\theta_N}(f_j)] ,$$
$$\underline{v}_{\theta_n}(f) = [e^{-j2\pi f\tau_1(\theta_n)}, e^{-j2\pi f\tau_2(\theta_n)}, \ldots, e^{-j2\pi f\tau_M(\theta_n)}]^T$$

Here $X_m(f_j)$, $S_n(f_j)$ and $N_m(f_j)$ are the Discrete Fourier coefficients of $x_m(t)$, $s_n(t)$ and $n_m(t)$, respectively, at the frequency f_j. A is the $M \times N$ direction matrix, and the vectors $\underline{v}_\theta(f)$ are referred to as the steering vectors of the array. We assume that the observation time is large so the time-bandwidth product is large. Under this assumption the Fourier coefficients associated with different frequencies are uncorrelated and the probability density function (*pdf*) of the received data is given by:

$$p(\mathbf{X}) = \prod_{l=1}^{L} p(\underline{X}_l) \tag{3}$$

where $\frac{1}{\sqrt{T}}\underline{X}_l$ (T is the total observation time), is an M-dimensional, zero-mean, complex Gaussian vector with covariance: $R_l = \eta_l I_M + A_l R_{sl} A_l^*$, $l = 1, \ldots, L$. I_M is the M-dimensional identity matrix, η_l is the spectrum of the additive noise process at each of the M sensors and R_{sl} is the N-dimensional cross-spectral matrix of the radiating sources.

2.2. On the Barankin bound

Denoting an unbiased estimate of a p-dimensional unknown parameter vector, $\underline{\phi}$, by $\hat{\underline{\phi}}$, the following matrix inequality holds [10]:

$$cov(\hat{\underline{\phi}}) \geq BB(\underline{\phi}) \geq BR(\underline{\phi}, \underline{\phi}_1, \underline{\phi}_2, \ldots) \geq CRB(\underline{\phi}) \tag{4}$$

where $cov(\cdot)$ stands for the covariance matrix, CRB and BB are the Cramer-Rao matrix and the Barankin matrix, respectively, and

$$BR(\underline{\phi}, \underline{\phi}_1, \underline{\phi}_2, \ldots) = T(\underline{\phi}, \underline{\phi}_1, \underline{\phi}_2, \ldots) D(\underline{\phi}, \underline{\phi}_1, \underline{\phi}_2, \ldots)^{-1} T^T(\underline{\phi}, \underline{\phi}_1, \underline{\phi}_2, \ldots) \tag{5}$$

The matrix T is given by: $T = [0, \underline{\phi}_1 - \underline{\phi}, \underline{\phi}_1 - \underline{\phi}, \ldots]$, where $\underline{\phi}_i$, $i = 1, 2, \ldots$, is a test point in the parameter vector space. D is a symmetrical matrix given by $\begin{bmatrix} 1 & \underline{1}^T \\ \underline{1} & B \end{bmatrix}$ where $\underline{1}$ is a vector of all ones and B is a matrix given by:

$$B_{ij} = E\left\{\frac{p(\mathbf{X}, \underline{\phi}_i)}{p(\mathbf{X}, \underline{\phi})} \frac{p(\mathbf{X}, \underline{\phi}_j)}{p(\mathbf{X}, \underline{\phi})}\right\} \tag{6}$$

The second inequality in (4) holds for any choice of test points (not less than p, the number of the unknown parameters). The Barankin bound matrix BB is derived for the set of test points which maximizes the matrix $BR(\underline{\phi}, \underline{\phi}_1, \underline{\phi}_2, \ldots)$. In our derivation we restrict ourself to p test points, each given by: $\underline{\phi}_i = [\phi_1, \phi_2, \ldots, \phi_i + \Delta\phi_i, \ldots, \phi_p]$. That is, the matrix T is determined by $\Delta\phi_1, \ldots, \Delta\phi_p$ which are chosen to maximize $BR(\underline{\phi}, \Delta\phi_1, \ldots, \Delta\phi_p)$. The resulted bound is not necessarily the greatest Barankin-type lower bound (since by adding more test points BR may be larger), but we argue that the threshold SNR set by our bound is practically unchanged when more test points are used.

The multidimensional version of the Barankin bound of (4)-(6) is applicable for data $\mathbf{X}$ of any distribution. For our problem, where $\mathbf{X}$ consists of LM-dimensional zero-mean complex Gaussian vectors, it can be shown [11] that (6) is given by:

$$B_{ij} = \prod_{l=1}^{L} \frac{det(R(f_l, \underline{\phi}))}{det(R(f_l, \underline{\phi}_i) det(R(f_l, \underline{\phi}_j))} \cdot \frac{1}{det(R^{-1}(f_l, \underline{\phi}_i) + R^{-1}(f_l, \underline{\phi}_j) - R^{-1}(f_l, \underline{\phi})} \tag{7}$$

where $R(f_l, \underline{\phi}_i)$ is the covariance matrix of the data Fourier coefficients at frequency f_l, evaluated with the assumed parameter vector $\underline{\phi}_i$. $\underline{\phi}$ is the true parameter vector. (7) holds as long as $R^{-1}(f_l, \underline{\phi}_i) + R^{-1}(f_l, \underline{\phi}_j) - R^{-1}(f_l, \underline{\phi})$, $i, j = 1, \ldots, p$, $l = 1, \ldots, L$ is a positive definite matrix. Once the best test points are chosen, physical conditions which cause $det(R^{-1}(f_l, \underline{\phi}_i) + R^{-1}(f_l, \underline{\phi}_j) - R^{-1}(f_l, \underline{\phi})) = 0$ for any $i, j = 1, \ldots, p$, $l = 1, \ldots, L$, *characterize a threshold phenomenon*. We find the threshold SNR for a single frequency. Based on the observation above, we argue that the threshold SNR in the multi-frequencies case can not be higher than the one evaluated for a single frequency. Moreover, if the signal is narrowband (in the sense that $\frac{ap}{\lambda}\frac{W}{\omega_0} << 1$, where ap is the array aperture, W is the processing bandwidth, ω_0 is the center frequency and λ is the corresponding wavelength) then the threshold is practically unchanged.

The unknown parameters vector in our problem consists unknown $DOAs$ and/or spectral parameters. Following similar argumentation, the threshold with more than one source cannot be at lower SNR than the one derived for a single source. Also, if sources' spectral parameters are unknown the threshold can only move to a higher SNR, relative to the case of known spectral parameters. In [9] we present formal proofs and specific results for the different cases we have studied. Here we focus on specific results and discuss them.

The mathematical definition of the threshold, as the SNR for which (6) becomes infinite, need not necessarily characterize the transition from the large errors regime, where the *mse* performance is dominated by ambiguity problems or by interferences, to the small-errors regime where achievable performance closely approaches the CRB. Actually, following the detailed study in [9], it appears that our definition characterizes the above mentioned physically meaningful threshold SNR in problems involving "moderate" values (of order 1) of the time-bandwidth product (TBP), where the available data concentrates in a single Fourier coefficient. As the time bandwidth product becomes larger, the Barankin bound decreases faster than the CRB and the two bounds met in SNR smaller than the one determined by our definition of the threshold. Yet, the "best" test points devised by the single frequency analysis are still appropriate for the case of large TBP and the qualitative discussion for the performance characteristic of well separated and closely spaced sources is still valid.

3. THRESHOLD SNR IN PASSIVE BEARING ESTIMATION - RESULTS

3.1. A single source

For $N = 1$ - a single source in spatially white noise - the SNR for which $det(R^{-1}(f_0, \underline{\phi}_i)+R^{-1}(f_0, \underline{\phi}_j)-R^{-1}(f_0, \underline{\phi})) = 0$, i.e., the threshold SNR for a single tone at frequency f_0 is:

$$r_{th1} = \frac{S(\omega_0)}{\eta(\omega_0)} = \frac{1}{M}\frac{1+\sqrt{1+8(1-mBP(\theta,\omega_0))}}{4(1-mBP(\theta,\omega_0))} \quad (8)$$

Where $mBP(\theta, \omega_0)$ is the relative value of the highest sidelobe of the array beampattern, when is aligned toward the source at bearing θ:

$$mBP(\theta,\omega_0) == \max_{\alpha \subset Q} |\frac{\underline{v}*(\theta,\omega_0)\underline{v}(\alpha,\omega_0)}{M}|^2.$$

$BP(\theta, \alpha, \omega_0) = |\frac{\underline{v}*(\theta,\omega_0)\underline{v}(\alpha,\omega_0)}{M}|^2$ is the conventional array beampatternd and Q is the set of all possible DOAs outside the mainlobe of the array beampattern. Since the normalized array beampattern, BP, is non-negative and not larger than unity, the threshold value exceeds M^{-1} in any given array geometry and its value is determined by the largest sidelobe of the array beampattern. For a linear uniform array this sidelobe is about -13dB and the threshold SNR is well approximated by M^{-1}, while theoretically this threshold SNR is achieved only with an ideal array beampattern having no sidelobes.

The threshold SNR of (8) is the same if the source spectral level is known or not [4],[9]. Also, the choice of the best test point for θ, at the largest sidelobe of the array beampatttern, agrees with the ambiguity analysis of this problem [6].

3.2. Two sources

The best test-angle associated with the interference-dominated regime in the case where the bearing of one source is estimated in the presence of another, interfering source appears to be the bearing of the other source. Likewise, the Kullback-Leibler distance [6] between the true and the mistaken models tends to get a local minimum around this test angle. This choice of this test angle yields the threshold SNR for the first source (at bearing θ_1):

$$r_{th2} = \frac{1}{4M}\{\frac{1}{1-BP(\theta_1,\theta_2,\omega_0)} + 2Mr_2 \quad (9)$$
$$+\sqrt{[\frac{1}{1-BP(\theta_1,\theta_2,\omega_0)})^2\frac{8+12Mr_2}{1-BP(\theta_1,\theta_2,\omega_0)} + 4M^2r_2^2]\}$$

where $r_2 = \frac{S_2(\omega_0)}{\eta}$ is the SNR of the other source (interference) and θ_2 is its bearing. The threshold SNR depends on the relative separation of the sources via the normalized array beampattern (the larger the value of BP at the interference direction is, the larger is the threshold SNR) and on the SNR of the interfering source. From (9) we have that $r_{th2} \geq \frac{1}{M} + r_2$ with equality if the interfering source lays in a null of the array beampattern, when aligned towards the source. However, when the two sources are well separated (not an 'high resolution' scenario) and the array geometry is reasonable, $\frac{1}{M} + r_2$ is a good approximation for the threshold SNR. Note also that the threshold value is symmetric with respect to the two sources.

The threshold SNR of (9) is observed only when the interference is "sufficiently strong", so $r_{th2} > r_{th1}$. Otherwise, r_{th1} is the observable threshold. If $r_{th2} > r_{th1}$ then for $r_{th2} > r > r_{th1}$ an interference dominated regime makes the mse performance of the bearing estimate of θ_1 exceeds the CRB by large factors.

In high resolution scenario the choice of a test point for θ_1 at θ_2 yields a CRB-type test point. Thus, the threshold tends to degenerate, except for very strong interference. That is, our analysis above is still valid, but since the CRB at high resolution is very high, the difference between 'small' and 'large' estimation errors is no longer significant, so the threshold effect is unclear. Adding more test points to the Barankin bound may make it clearer, but not significantly.

Also for the case of two sources we show [9] that the availability of prior knowledge of the interference bearing and/or of the sources spectra affect the threshold SNR insignificantly, at least in the single frequency analysis.

3.3. Numerical examples

The results of the numerical examples are depicted in Figs. 1a and 1b. The array used in the example is a 6-elements equally spaced linear array. The distance between adjacent sensors is $\lambda/2$. The scenario assumed a radiating source at the broadside of the array of known spectral level. Its DOA is estimated in the presence of an interference. We assume that the interference is either well-separated from the source (separation of 40^0, or closely-spaced (separation of 3^0). All parameters of the interference (DOA and spectral level) are assumed known. For each of the two cases we plot the Barankin bound (solid line) and the CRB (dashed line) as a function of the source SNR for the case where the interference is strong (interference to noise ration - INR=5db) and weak (INR=0db). The lowest solid line in both figures is the Barankin bound for a single source - where no interference is present.

When the sources are well separated (Fig. 1a), the CRB associated with weak or strong interference are practically equal. The threshold SNR, in which the BB and the CRB coincides is clear. This point is exactly the one predicted by (8) for a single source and by (9) for the two sources scenario. The threshold SNR increases with the interference SNR r_2, as predicted by the theory. In Fig. 1b, where the sources are spatially close, the threshold phenomenon is unclear. Mathematically, the Barankin bound still exceeds the CRB in very large SNR values though its improvement is insignificant.

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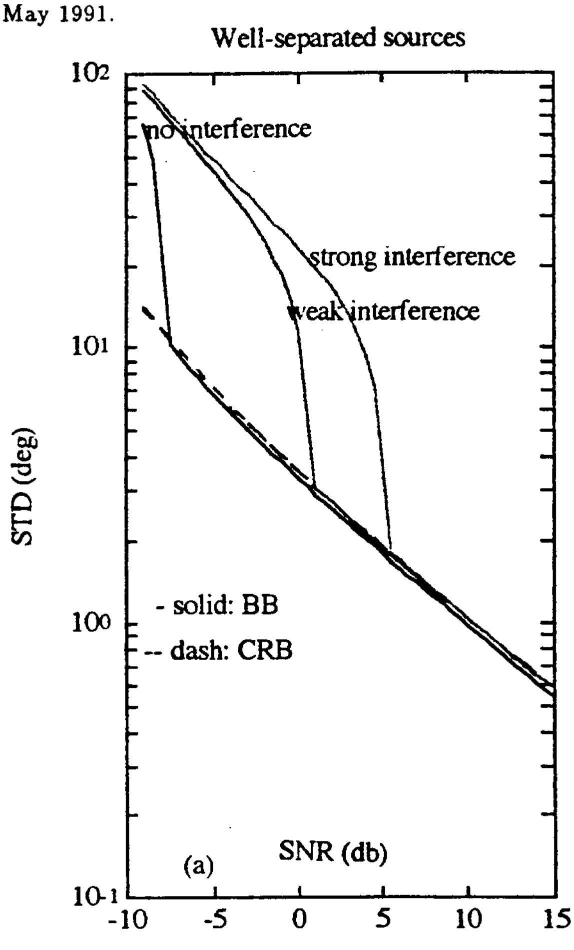

Fig. 1a: The Barankin bounds (solid lines) and the *CRBs* (dash/dot) on the bearing estimation of a far-field narrowband source in the presence of a well-separated interference.

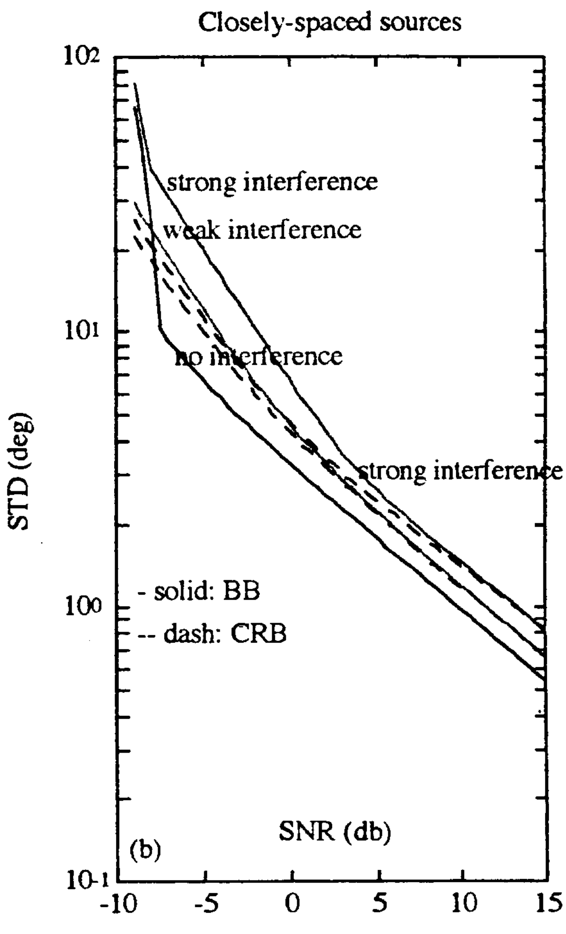

Fig. 1b: The Barankin bounds (solid lines) and the *CRBs* (dash/dot) on the bearing estimation of a far-field narrowband source in the presence of a closely-spaced interference.