A PARAMETRIC SET OF SPECTRAL ESTIMATES AND THEIR PERFORMANCE

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ABSTRACT

A novel rigorous approach to the spectral density estimation problem based on the trigonometric moment problem technique is considered. Using the trigonometric moment problem results, all possible extrapolations of the autocorrelation function, which are in agreement with a set of known values are found. A wide set of spectral estimators is described in terms of polynomials orthogonal with respect to the given autocorrelation sequence. The parametric representation for this set is given. The performance of the proposed spectral estimator with arbitrary parametrization function is established.

1. INTRODUCTION

At present, there are several standard methods of dealing with the classical problem in digital signal and sensor array processing. [1, 2, 3]. These methods include the conventional nonparametric Fourier approach and more recently developed parametric techniques such as maximum entropy (ME), maximum likelihood (ML), MUSIC, ESPRIT and others.

Until 1967, most of the procedures used for estimating the spectral density of a stochastic process were based on the Blackman-Tukey approach [1]. The expectation of an estimate is equivalent to the convolution of the true spectral density of the stochastic process with the spectral window. The statistical stability and resolution of the spectral density estimate using the Blackman-Tukey procedure are highly dependent on the choice of the window function [4]. Moreover, the spectral density estimator based on Blackman-Tukey approach is linear because it involves the use of linear operations on the available time series. A major problem with the Fourier transform is that the accuracy of the mode estimates is roughly inversely proportional to the total simulation time interval. Moreover, there is problem with appropriate choice of the windowing function.

The windowing problem may be overcome by using the nonlinear estimators of spectral density based on ME approach, ML approach or others. But these approaches are not universal. Really, these approaches do not permit one to get other spectral estimations which corresponds to time series, whose autocorrelation functions agree with the same set of known values, i.e. to get other extrapolations of the correlation function.

Besides, when the spectral estimate with good resolution is necessary to solve a more concrete problem, for example, the power spectral density estimation problem with additional *a priori* information about the spectrum, it seems unreasonable to apply the known methods of spectral estimation, such as ME method, ML method or others in their original form, because these methods of spectral estimation are not intended to solve concrete problems [5].

So, in spite of the progress in spectral estimation and a list of successful practical application of the more popular methods of spectral estimation such us ME, MUSIC, ML and some others, new methods for effective spectral estimation are necessary. Besides, it is desired to have a common approach both for the construction of spectral estimation methods and for the investigation of their performance.

The aim of this paper is to propose and develop a novel rigorous approach to spectral density estimation problem. We describe all possible extrapolations of the autocorrelation function which agree with a set of known values. The set of possible spectral estimations is described as a parametric family of spectral estimators with parametrization function. The quantitative performance of the proposed spectral estimator's family is established.

2. SPECTRAL DENSITY ESTIMATION PROBLEM

2.1. Problem formulation

Consider the problem of estimating a power spectral density function given only that it is positive on the spectral support, zero outside and compatible with the known values of the autocorrelation sequence.

The aim of spectral estimation is to find an estimate of the spectrum which agrees with the set of the known values of autocorrelation sequence c_k, $k = 1, 2, \dots, N$, i.e., spectral density estimation problem is to determine the nonnegative function $P(\theta)$ such that

$$P(\theta) \geq 0, \quad \text{for} \quad \theta \in [-\pi, \pi], \tag{1}$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(\theta) e^{-ik\theta} \, d\theta = c_k, \quad k = 0, 1, \dots, N. \tag{2}$$

Note that the considered problem (1)–(2) may be unsolvable for an arbitrary sequence of the values c_k if we strictly follow conditions (2). To become solvable, this problem has to be regularized.

There are two ways to attach a meaning to the problem (1)–(2). The first way is to replace conditions (2) by approximate relations. In this case the problem arises of choosing a good approximate relation. Another way to attach a meaning to the problem (1)–(2) is to relax the condition (1) and replace it by $P(\theta) \geq -\mu$, for $\theta \in [-\pi, \pi]$, where μ is a positive value. Then, the shifted spectral estimation $P_\mu(\theta) = P(\theta) + \mu$ will be nonnegative and problem transforms to problem (1)–(2) with modified autocorrelation sequence $c_k + \mu\delta_{k0}$, $k = 0, 1, \ldots, N$, where δ_{k0} is the Kroneker symbol.

It is relevant to remark that the problem for shifted function $P_\mu(\theta)$ is not solvable for arbitrary positive value μ. However, we can always make the problem solvable by choosing a sufficiently large value μ.

The appearing item $\mu\delta_{k0}$ can be explained in a different way. First of all, the measurement data really may have a white noise with dispersion characteristic μ. Besides, by having added a component $\mu\delta_{k0}$ to regular autocorrelation coefficients c_k we imply an obvious method to impart regularity to the considered problem. In this way the value μ is a regularization parameter. Obviously, the regularized problem is equivalent to initial spectral density estimation problem (1)–(2) for new autocorrelation sequence $c_k + \mu\delta_{k0}$, $k = 0, 1, \ldots, N$. Therefore, below we consider the spectral density estimation problem (1)–(2) with the regularization parameter μ, whenever the initial spectral density estimation problem (1)–(2) for sequence c_k, $k = 0, 1, \ldots, N$ is unsolvable.

2.2. Extrapolation of autocorrelation sequence

Let us now assume, that a solution of the spectral density estimation problem (1)–(2) is known. Then we may calculate the unknown terms of the autocorrelation sequence, i.e, c_k for all $k \geq N$ and so to extrapolate the autocorrelation sequence. The inverse is also true, i.e., if we have found an extrapolated autocorrelation sequence $\hat{c}_k$, $k = 0, 1, \ldots$ such that its $N + 1$ first terms are equal to the known values of the autocorrelation sequence, i.e., the values

$$\hat{c}_k = c_k, \qquad \text{for} \quad k = 0, 1, \ldots, N,$$

then the spectral estimate can be defined by means of the inverse Fourier transform

$$\hat{P}(\theta) = \sum_{k=-\infty}^{\infty} \hat{c}_k e^{ik\theta}, \tag{3}$$

where $\hat{c}_{-k} = \bar{\hat{c}}_k$ for all integers k.

Let us note, that the extrapolated autocorrelation sequence $\hat{c}_k$ can be chosen arbitrary provided that the function $\hat{P}(\theta)$ is nonnegative. The last condition gives us a restriction for the possible extrapolations of the autocorrelation sequence c_k, $k = 0, 1, \ldots, N$. Namely the autocorrelation sequence c_k, $k = 0, 1, \ldots, N$ and the extrapolated autocorrelation sequences $\hat{c}_k$ $k = 0, 1, 2, \ldots$ have to be positive-definite [6, 7]. The set of all spectral estimations written in form (3) with an arbitrary positive definite extrapolated sequence $\hat{c}_k$, $k = 0, 1, 2, \ldots$ forms the Carateodory class of function [8]. The Catateodory class is closely associated with the trigonometric moment problem [6], which plays an important part for the description of all solutions of the spectral density estimation problem.

2.3. The trigonometric moment problem

The trigonometric moment problem consists of a collection of complex numbers

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\theta) e^{ik\theta}\, d\theta, \qquad k = 0, 1, \ldots, N, \tag{4}$$

where $P(\theta)$ is an unknown positive function of θ. Assuming that the moments c_k, $k = 0, 1, \ldots, N$ of $P(\theta)$ are exactly known, the problem is to determine the function $P(\theta)$. Let us note, not any set of numbers c_k, $k = 0, 1, \ldots, N$ can be the moments of nonnegative function, i.e., the trigonometric moment problem for an arbitrary sequence of moments c_k, $k = 0, 1, \ldots, N$ can be unsolvable. We will assume that the Toeplitz determinants $\Delta_n = \det \|c_{n-k}\|$ are nonzero for all integer $n = 0\ 1, \ldots, N$.

Then there exists the system $\{P_n(z)\}$ of polynomials of the first type and the system $\{P_n(z)\}$ of polynomials of the second type orthogonal with respect to the unknown measure $P(\theta)$, which correspond to given moment sequence $\{c_k\}_{k=0}^{N}$ [6]. The monic polynomials $\{P_n(z)\}$ satisfy the recurrence relations for the orthogonal sets of polynomials:

$$P_{n+1}(z) = zP_n(z) + b_{n+1} P_n^*(z), \tag{5}$$

where $R^*(z) \stackrel{\text{def}}{=} z^k \overline{R}(1/z)$ for any polynomial $R(z)$ of degree k; b_k are some constants for which formal analytic relations in determinant form can be obtained.

The following criterion of solvability of the moment problem (4) holds [9]:

Theorem 1 *The trigonometric moment problem (4) is solvable if and only if the inequalities*

$$|a_k| < 1, \quad k = 0, 1, \ldots, N,$$

hold, where $\{a_k\}_{k=0}^{N}$ are the coefficients of the recurrence relation of polynomials orthogonal with respect to the moment sequences $\{c_k\}_{k=0}^{N}$.

To attach this theorem a constructive mean, a recurrence algorithms, for example, a Levinson algorithm has to be used to calculate the coefficients a_k of the recurrence relation for orthogonal polynomials [9].

Let us now assume that the necessary conditions of solvability of the trigonometric moment problem are true. Then all functions $P(\theta)$, for which the relation (4) is fulfilled for all $k = 0, 1, \ldots, N$, can be described by means of the linear-fractional transformation [6].

Theorem 2 *Let $P_N(z)$ and $Q_N(z)$ be the monic polynomials of first and second type orthogonal with respect to given moment sequence $\{c_k\}_{k=0}^{N}$. Let the moment problem (4) be indeterminate. Then all solutions of the moment problem*

(4) can be described by means of the linear-fractional transformation

$$P_\varepsilon(z) = \frac{c_0}{h_N} \cdot \Re e \frac{zQ_N(z)\varepsilon(z) + Q_N^*(z)}{zP_N(z)\varepsilon(z) - P_N^*(z)}, \quad z = e^{i\theta} \qquad (6)$$

of the function $\varepsilon(z) \in B$, where B is a class of functions holomorphic in $\mathbb{C}$ and bounded there in modulus by 1; $P_n^*(z) = z^n\overline{P}_n(1/z)$, $Q_n^*(z) = z^n\overline{Q}_n(1/z)$, $h_n = \frac{\Delta_n}{\Delta_{n-1}}$.

Due to Theorem 2 any solution $P(\theta)$ of the moment problem can be expressed by means of the linear-fractional transformation (6) with the holomorphic function $\varepsilon(z)$ corresponding to this solution.

Using the properties of the orthogonal polynomials [6] it is easy to transform (6) to form

$$P_\varepsilon(z) = \frac{1 - |\varepsilon(z)|^2}{|zP_N(z)\varepsilon(z) - P_N^*(z)|^2}, \quad z = e^{i\theta}. \qquad (7)$$

3. SPECTRAL ESTIMATOR'S FAMILY

In this section we consider the spectral estimation problem as a trigonometric moment problem.

As it follows from the previous section any solution of the spectral estimation problem (1)–(2) for the regularized sequence $c_k + \mu\delta_{k0}$, $k = 0, 1, \ldots, N$ can be represented in form (7) where $\varepsilon(z)$ is a parametrization function corresponding to this spectral estimate, $P_N(z)$ is the polynomial with the coefficients satisfying to the Yule-Walker system and $P_N^*(z)$ is a polynomial conjugate to $P_N(z)$.

Using representation (7) and an effective algorithm for calculation of the orthogonal polynomial $P_N(z)$ and its conjugate by the given autocorrelation values one may obtain different spectral estimations. Obviously, the form and performance of a concrete spectral estimator depend on the choice of the parametrization function $\varepsilon(z)$. Using an appropriate parametrization function, the known spectral estimation can be obtained. For example, in the special case where $\varepsilon(z) \equiv 0$ relation (7) is simplified and passed to Burg's maximum entropy spectral estimation

$$P_{ME}(\theta) = |P_N(e^{i\theta})|^{-2},$$

where $P_N(z)$ is the polynomial with the coefficients satisfying to Yule-Walker system.

Another example is the maximum entropy solution of spectral density estimation problem with *a priori* information about spectrum [5, 10], where the aim of spectral estimation is to find the function which agrees with the set of the known values of autocorrelation function and given *a priori* information. In view of the given *a priori* information, the original spectral estimation problem reduces to spectral estimation problem with gaps in spectrum [10]. The spectral estimation with desired properties, can be found using relation (7). However, to find the solution of this spectral estimation problem, we have to choose the parametrization function $\varepsilon(z)$.

The algorithm choosing the parametrization function $\varepsilon(z)$ to construct the maximum entropy solution of the spectral estimation problem with *a priori* information about the spectrum can be found in [5, 10].

Let us now explain the way of choosing of the parametrization function $\varepsilon(z)$ to get the linear spectral estimator.

Obviously, any linear spectral estimate can be represented as a weight Fourier transform from the known correlation coefficients c_k

$$P(z) = 2\Re e \sum_{k=0}^{N} \rho_k c_k z^k, \quad z = e^{i\theta} \qquad (8)$$

where ρ_k is the weight coefficients, which are determined by the chosen windowing function. In the particular case $\rho_k = 0$ for all $k = 0, 1, \ldots, N$ the relation (8) transforms to ordinary Fourier spectral estimate.

From relations (6) and (8) we have

$$\varepsilon(z) = \frac{A_{2N}(z)}{zB_{2N}(z)}, \quad z = e^{i\theta}, \qquad (9)$$

where

$$A_{2N}(z) = \frac{h_N}{c_0} P_N^*(z) \sum_{k=0}^{N} c_k\rho_k z^k + Q_N^*(z)$$

and

$$B_{2N}(z) = \frac{h_N}{c_0} P_N(z) \sum_{k=0}^{N} c_k\rho_k z^k - Q_N(z)$$

are polynomials of power $2N$. The coefficients of this polynomials can be easily expressed through the initial correlation coefficients c_k and the given weights ρ_k.

From relation (9) it follows that the parametrization function ε is determined by the autocorrelation coefficients c_k and windowing function in unique way. Using varies windowing functions one can get any known linear spectral estimate. Thus, relations (7) and (9) give us a description of a class of linear spectral estimates by means of linear fractional representation.

4. PERFORMANCE OF SPECTRAL ESTIMATOR WITH ARBITRARY PARAMETRIZATION FUNCTION

In this section we study the properties of the spectral estimator in form (7) with an arbitrary parametrization function $\varepsilon(z)$.

As it follows from relation (7) the spectral estimation $P_\varepsilon(\theta)$ with an arbitrary function $\varepsilon(z)$ from the class B is nonnegative for all $\theta \in [-\pi, \pi]$. Moreover, due to theorem 2 the function $P_\varepsilon(\theta)$ defined by (7) satisfies the relation (1), i.e., the relation (1) is fulfilled for arbitrary parametrization function $\varepsilon(z)$ bounded in modulus by 1.

Having investigated the performances of Burg's method, many researchers discovered that the spectral resolution of Burg's algorithm is indeed higher than that of the classical methods based on the discrete Fourier transform.

Let us consider performances of the spectral estimator (7) with an arbitrary function $\varepsilon(z)$.

To take into consideration quantitative characteristic for the proposed method we consider the model example of one spectral line with coordinate θ_0 and power p in white noise with dispersion characteristic ϵ and study the behavior of the spectral estimation $P_\epsilon(\theta)$ in a neighbourhood of the point $\theta = -\theta_0$.

Let $\tilde{c}_k = pe^{-ik\theta_0} + \epsilon\delta_{k0}$ be a given model autocorrelation sequence. Using the Levinson algorithm [11] to solve the Yule-Walker system the polynomial $P_N(z)$ and its conjugate are found

$$P_N(e^{i\theta}) = e^{iN\theta}\left(1 + \frac{1}{N(1+\alpha)}\frac{e^{-iN(\theta+\theta_0)}-1}{e^{i(\theta+\theta_0)}-1}\right) \quad (10)$$

$$P_N^*(e^{i\theta}) = \left(1 + \frac{1}{N(1+\alpha)}\frac{e^{iN(\theta+\theta_0)}-1}{e^{-i(\theta+\theta_0)}-1}\right), \quad (11)$$

where $\alpha = \epsilon/Np$.

Substituting the expression for $P_N(z)$ and for $P_N^*(z)$ into (7) and making some transformations, we find the expression for spectral estimate with arbitrary parametrization function in a neighbourhood of the point $\theta = -\theta_0$

$$P_\epsilon(\theta) \simeq (1+\alpha)^2[(A(\theta)+1)^2 + B^2(\theta)][1 - |\varepsilon(e^{i\theta})|^2] \times \left[\left(\frac{N+1}{2}[(A(\theta)+1)^2 + B^2(\theta)](\theta+\theta_0) + 2\alpha B(\theta)\right)^2 + \alpha^2\left((A(\theta)-1)^2 + B^2(\theta)\right)^2\right]^{-1}, \quad (12)$$

where $\alpha = \epsilon/Np$ and $A(\theta)$ and $B(\theta)$ are real and image parts of the function $e^{i(N+1)\theta}\varepsilon(e^{i\theta})$.

Let us now assume the functions $A(\theta)$, $B(\theta)$ and $|\varepsilon(e^{i\theta})|$ are weakly changing in a neighbourhood of the point $\theta = -\theta_0$. Then using relation (12) and this assumption the width at half-power points for spectral estimate restored by means of the proposed technique is obtained

$$\Delta\theta_{0,5} \simeq \frac{4\alpha}{N+1}C, \quad (13)$$

where

$$C = \frac{\sqrt{(A(\theta_0)-1)^2 + 9B^2(\theta_0)}}{(A(\theta_0)+1)^2 + B^2(\theta_0)}. \quad (14)$$

The relations (13)–(14) characterize the spectral estimator in form (7) with arbitrary parametrization function $\varepsilon(z)$. Using this relation a performance for spectral estimator with concrete parametrization function can be obtained.

For example, in a particular case where $\varepsilon(z) \equiv 0$ our assumption is true and in this case $C = 1$. Substituting $C = 1$ into (13) we have a performance of Burg's maximum entropy spectral estimator. The width at half-power points of spectral estimation restored by means of maximum entropy method is

$$\Delta\theta_{0,5}^{ME} \simeq \frac{4\alpha}{N+1}$$

The relations (13) and (14) allow us to compare the spectral estimator in form (7) with the chosen parametrization function $\varepsilon(z)$ and Burg's maximum entropy spectral estimation.

Note, that the value C depends on parametrization function $\varepsilon(z)$ and can be in each particular case both larger (for example, if $A(\theta_0) = 0$, $B(\theta_0) = 0.2$) and smaller (for example, if $A(\theta_0) = 1$, $B(\theta_0) \simeq 0.2$) than 1. So, the family of spectral estimators in form (7) has estimators with performance both better and worse than that of Burg's method.

Moreover, from relations (13) it follows that the restoration performance improves with a decrease of the regularization parameter ϵ and with an increase in the number N of autocorrelation coefficients and the spectral line power p.

CONCLUSION

We have considered a novel approach to the spectral density estimation problem based on the moment method technique. Using this approach all possible extrapolations of the autocorrelation function which are in agreement with a set of known values were found. A wide spectral estimator's family has been obtained in terms of polynomials orthogonal with respect to a given autocorrelation sequence. The performance of the proposed spectral estimator with an arbitrary parametrization function has been investigated.

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REFERENCES

[1] R.B. Blackman and J.W. Tukey. *The measurement of power spectra from the point of view of communications engineering.* Dover, New York, 1959.

[2] J.P. Burg. Maximum entropy spectral analysis. In *Proc. 37th. Meeting Soc. Exploration Geophysicists*, Oklahoma City, OK, 1967.

[3] J. Capon. *Nonlinear Methods of Spectral Analysis, Ed.S. Haykin*, volume 34 of *Topics in Applied Physics*, chapter Maximum Likelihood Spectral Estimation, pages 155–179. Springer-Verlag, 1983.

[4] G.M. Jenkins and D.G. Watts. *Spectral Analysis and its Applications.* Holden-Day, San Francisco, 1969.

[5] R. Ugrinovsky. Maximum entropy algorithm for spectral estimation problem with a priori information. *J. Math. Phys.*, 35:4372, 1994.

[6] N.I. Achieser. *The Classical Moment Problem and Some Related Questions in Analysis.* Oliver & Boyd, Edinburgh, 1965.

[7] H.J. Landau. Maximum entropy and the moment problem. *Bulletin of the American Mathematical Society*, 16:47–77, 1987.

[8] C. Carathéodory. Über den variabilitätsbereich der fourierschen konstanten von positiven harmonischen funktionen. *Rend. Palermo*, 32:193–217, 1911.

[9] S.A. Korzh, I.E. Ovčarenko, and R.A. Ugrinovsky. Chebyshev recursion — some analytical, computational and applied aspects. *Ukrainskii Matematicheskii Zhurnal*, 45:625–646, 1993. (English translation in: Ukrainian Mathematical Journal, 1993, 45).

[10] R. Ugrinovsky. Maximum entropy algorithm for spectral estimation problem with gaps. In *ICASSP'94: Proc. Int. Conf. Acoust. Speech Sign. Proc.*, volume IV, pages 453–456, Adelaide, 19–22 April 1994.

[11] N. Levinson. The wiener rms (root-mean-square) error criterion in filter design and prediction. *J. Math. Physics*, 25:261–278, 1947.