A MULTIPLE WINDOW METHOD FOR ESTIMATION OF A PEAKED SPECTRUM

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ABSTRACT

This paper proposes a new multiple window method for estimating a peaked spectrum. The multiple windows are adapted to the signal, giving a less biased estimate for estimation of peaks than does the Thomson Multiple Window method. Still the result from estimation of a flat spectrum shows comparable results in variance reduction.

The method is based on solution of an eigenvalue problem where the eigenvectors of a special correlation matrix are used as multiple windows. The correlation matrix corresponds to a low frequency dominant spectrum with limited bandwidth. The design results in windows that are further improved by a penalty function to reduce leakage from nearby frequencies. This gives a better estimate when the process contains of large spectrum dynamics.

1. INTRODUCTION

The problem of estimating a spectrum from a finite (small) number of samples of a stationary process is well known. The main problems are to achieve a good resolution and a low variance in the estimate. Among the non-parametric methods, the Multiple Window method by Thomson [1] should be mentioned. It is designed for white noise and is consequently preferred for smooth spectrum or spectrum that is constant in a limited frequency range. However, when the spectrum contains peaks, the resolution of these is limited to the bandwidth decided in the window design. A better way is then to find windows that describe spectrum including peaks. By choosing a low frequency dominant peaked spectrum for the window design, the bias in the peaks is reduced, compared to the Thomson Windows.

2. PROBLEM FORMULATION

One way to express spectrum estimation is to give it a bandpass filter interpretation, [2]. This can be seen as a frequency shift of f_0 followed by a lowpass filter, see Figure 1. The variance estimate from the output of the filter gives the estimate of the spectrum, $\hat{S}_x(f_0)$, at the shifted frequency. The resolution is decided from the bandwidth of the filter $H(f)$.

The estimate can be written as

$$\hat{\sigma}_y^2 = \hat{S}_x(f_0) = \int_{-B/2}^{B/2} |H(f)|^2 S_x(f - f_0) df \qquad (1)$$

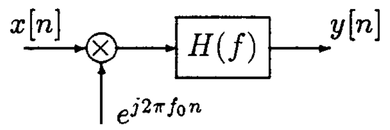

Figure 1: Spectrum estimation expressed as shift followed by a lowpass filter.

where B gives the resolution of the estimate. Under the condition that B is given, the goal is to maximize the power output from the filter. For $f_0 = 0$ Eq. (1) is rewritten as

$$\max \int_{-B/2}^{B/2} |H(f)|^2 S_x(f) df. \qquad (2)$$

Defining $H(f) = \mathbf{h}^T \phi(f)$ with $\mathbf{h} = [h(0)\ h(1)\ ...\ h(M-1)]$ and $\phi(f) = [1\ e^{-j2\pi f}\ ...\ e^{-j2\pi(M-1)f}]^T$ gives

$$\max \mathbf{h}^T \int_{-B/2}^{B/2} \phi(f) S_x(f) \phi^*(f) df\, \mathbf{h} =$$
$$\max \mathbf{h}^T \mathbf{R} \mathbf{h} \qquad (3)$$

where the $M \times M$ Toeplitz correlation matrix $\mathbf{R}$ has the elements $r(m_1, m_2) = r(l) = r_x(l) * \frac{B}{2\pi}\text{sinc}(\frac{B}{2}l)$ where $l = |m_2 - m_1|$. The maximized output is limited by the constraint that $\mathbf{h}^T\mathbf{h} = 1$. With the presumption that spectrum might include peaks, $S_x(f)$ is specified as a low frequency peaked spectrum which is bandlimited to $B/2$ as in Figure 2. In this paper this is done by calculating the covariance function for a low frequency dominating peaked spectrum from a polynomial function. The sharpness is decided by the order of the polynomial and the resolution by a box multiplying the polynomial function.

3. SOLUTION

The solution of Eq.(3) is given by the corresponding eigenvector to the largest eigenvalue by

$$\mathbf{R}\mathbf{q}_i = \lambda_i \mathbf{q}_i \quad i = 1 \ldots \mathrm{M}. \qquad (4)$$

To find the multiple windows, the optimization problem is reformulated as

$$\max\ \text{trace}\ \mathbf{Q}^T\mathbf{R}\mathbf{Q}\ \text{subject to}\ \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \qquad (5)$$

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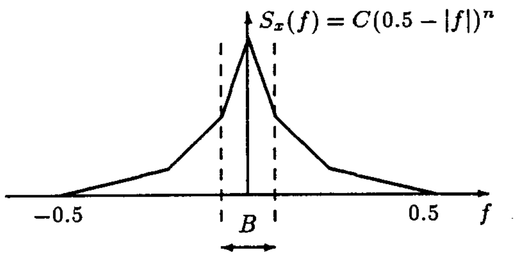

Figure 2: The cut low-frequency peaked spectrum where the sharpness of the peak is decided from a polynomial and the resolution from the box cutting function.

where $\mathbf{Q} = [\mathbf{q}_1\ \mathbf{q}_2\ ...\ \mathbf{q}_M]$. The multiple windows are given by the eigenvectors corresponding to the K largest eigenvalues where K<M. The spectrum estimate is given as

$$\hat{S}_x(f) = \frac{\sum_{i=1}^{K} \lambda_i |\mathbf{q}_i^T \mathbf{\Phi}(f)\mathbf{x}|^2}{\sum_{i=1}^{K} \lambda_i} = \sum_{i=1}^{K} \lambda_i^n |\mathbf{q}_i^T \mathbf{\Phi}(f)\mathbf{x}|^2 \quad (6)$$

where $\mathbf{x}$ is the sampled data vector of length M and $\mathbf{\Phi}(f)$ the diagonal matrix with $\phi(f) = [1\ e^{-j2\pi f} \ldots e^{-j2\pi f(M-1)}]$ at the diagonal. The new weighting factor λ_i^n is normalized, thus $\sum_{i=1}^{K} \lambda_i^n = 1$. The eigenvectors fulfil the requirement of being orthogonal and they also give a known bandlimited resolution. We call this method the Polynomial Multiple Window method (PolyMW).

However, the demand for low sidelobes to reduce the leakage from nearby frequencies have not been taken into consideration. One cause for large sidelobes is that the eigenvector starts and ends in a discontinuity. This gives a large contribution at all frequencies. To reduce the sidelobe leakage, a penalty function that compensates discontinuities, if they exist, has been introduced. The problem is then reformulated into

$$\max \text{ trace } \mathbf{Q}^T\mathbf{R}\mathbf{Q} \text{ subject to } \mathbf{Q}^T\mathbf{W}\mathbf{Q} = \mathbf{I} \quad (7)$$

where the diagonal penalty matrix $\mathbf{W}$ is defined with a '1/raised cosine' function at the diagonal. The solution is given by the eigenvectors and eigenvalues of the generalized eigenvalue problem

$$\mathbf{R}\mathbf{q}_i = \lambda_i \mathbf{W}\mathbf{q}_i \quad i = 1\ ...\ \text{M}. \quad (8)$$

This method is called the Weighted Polynomial Multiple Window method (W-PolyMW).

4. NUMERICAL EXAMPLES

To show both advantages and disadvantages of the proposed methods, three different examples of processes are investigated. The data contain 25 realizations of 128 data samples each. A picture of the variance is given as the 25 different estimates are plotted. The order of the polynomial function used for the window design, see Figure 2, is $n = 20$ for the two methods described in section 3. These methods are compared to a single window method with Hanning window (Hanning) to see the difference in variation (variance) between a multiple window and single window method. A comparison is also made to the Thomson Multiple Window method (ThomMW). For the three multiple window methods the resolution is set to $B = 0.08$. The eigenvectors that correspond to the K=8 largest eigenvalues are chosen as windows.

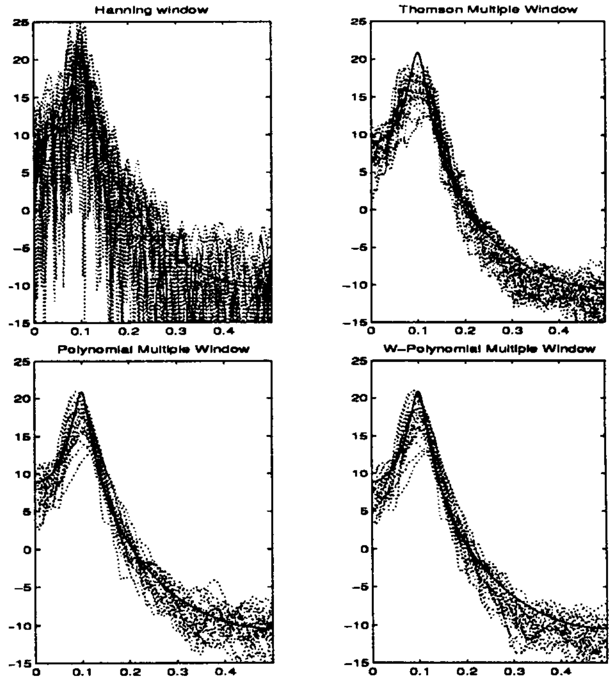

Figure 3: The spectrum estimates of the different methods in Example 1. True Spectrum: Solid line, Spectrum Estimates: Dotted lines.

Example 1

The PolyMW and W-PolyMW methods have the ability of estimating peaks in spectrum with less bias than e.g. the ThomMW method. This is shown with an AR-process of order 2 with poles in $0.92e^{\pm j2\pi 0.1}$. The true spectrum is represented by the solid line in Figure 3. The dotted lines are the spectrum estimates and their spread indicates the variance. The single Hanning window shows large variation compared to the multiple windows. The ThomMW method also shows the characteristic broadening of the peak that is given by the resolution of the windows. With the PolyMW and W-PolyMW methods, the peak is given as a less biased estimate. The choice of more signal adapted windows thereby results in a better estimate without loss in variance.

Example 2

In this example the process does not contain peaks, but is characterized by a lowpass filter. An ARMA-process of order 3 is generated by white noise filtered through a Butterworth filter with bandwidth $f = 0.1$. The result of Figure 4

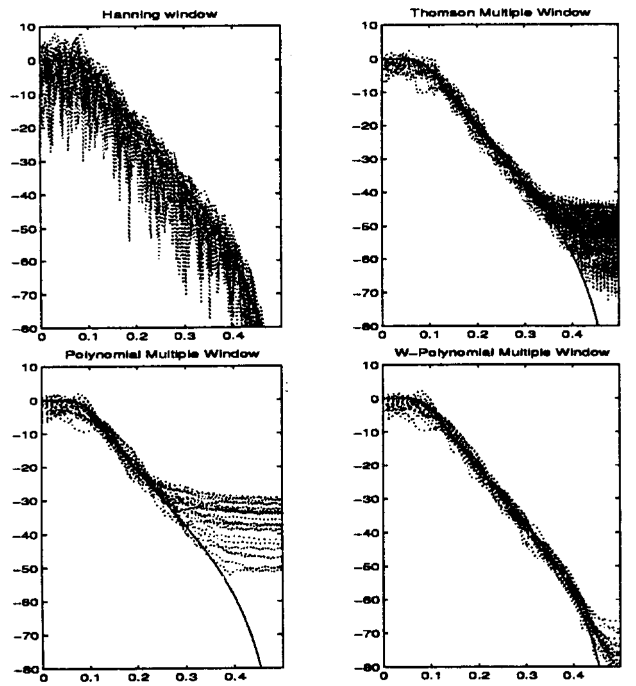

Figure 4: The spectrum estimates of the different methods in Example 2. True Spectrum: Solid line, Spectrum Estimates: Dotted lines.

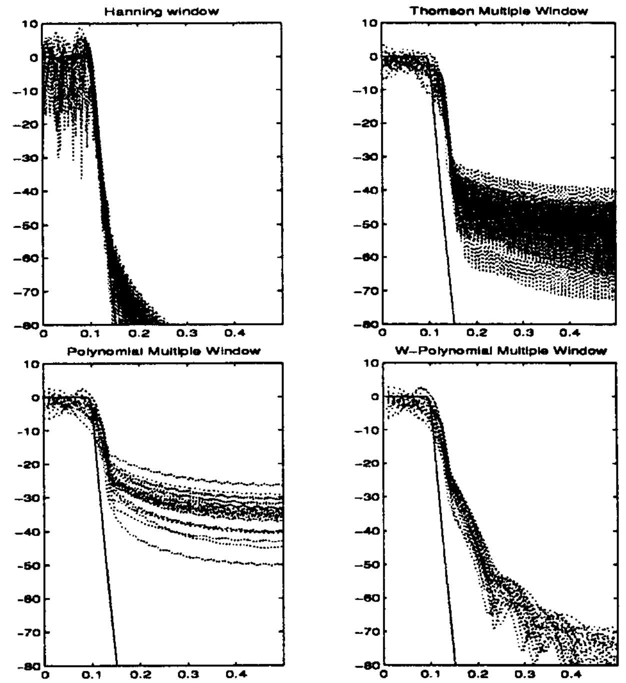

Figure 5: The spectrum estimates of the different methods in Example 3. True Spectrum: Solid line, Spectrum Estimates: Dotted lines.

indicates that the W-PolyMW method is outstanding when the variance is approximately the same for all multiple window methods and it has a better ability to reduce leakage in estimate in far-away low spectrum levels. However, it should not be forgotten that if the spectrum has very large dynamics, then the first window of the ThomMW method is preferable as the sidelobe leakage is very small, -140 dB. On the other hand, if the variance of the estimate is important, and we use 8 windows, the total sidelobe leakage is reduced to -30 dB for the Thom MW method. This is the same as the leakage for the windows of the PolyMW and W-PolyMW methods. The reason for the good estimates in Example 2 is that the W-PolyMW method has its first sidelobe at -30 dB but the level of further away sidelobes is lower. This is an advantage for estimation of a smoother spectrum as the one in Example 2.

Example 3

In Figure 5, an ARMA-process of order 20 made from a Butterworth filter with bandwidth $f = 0.1$ is shown. This example shows that sharp flanks are difficult to track. The resolution limits the ability. The conclusion is that the PolyMW and W-PolyMW methods are able to estimate sharp flanks with satisfactory result compared to the ThomMW method. However, again, if only the first window of the ThomMW method is used, the result will show large variance but an unbiased estimate for dynamics of up to 140 dB. The first windows of the PolyMW and W-PolyMW methods do not have this ability.

5. CALCULATION OF BIAS AND VARIANCE

The examples in section 4 shows that the multiple window methods are superior to the single window method. However, the behaviour of the different multiple window spectrum estimates are more or less the same. To see the difference between the methods, the bias and variance could be calculated with knowledge of the true spectrum. Bias is defined as

$$\text{Bias} = \frac{E[\hat{S}(f)] - S(f)}{S(f)} \tag{9}$$

where the expected value of the spectrum estimate is calculated as

$$E[\hat{S}(f)] = \frac{1}{K}\sum_{i=1}^{K} \lambda_i^n \mathbf{q}_i^T \mathbf{\Phi}^H(f)\mathbf{R}\mathbf{\Phi}(f)\mathbf{q}_i. \tag{10}$$

The orthogonality of the windows is not useful in the calculation of the variance as the process for which we estimate the spectrum is not white. The variance is then calculated as

$$\text{Variance} = \frac{\frac{1}{K^2}\sum_{j=1}^{K}\sum_{i=1}^{K} \text{cov}(\hat{S}_i(f)\hat{S}_j(f))}{S^2(f)} \tag{11}$$

where

$$\begin{aligned}\text{cov}(\hat{S}_i(f)\hat{S}_j(f)) = \lambda_i^n\lambda_j^n \quad & (|\mathbf{q}_i^T\mathbf{\Phi}^H(f)\mathbf{R}\mathbf{\Phi}(f)\mathbf{q}_j|^2 \\ & +|\mathbf{q}_i^T\mathbf{\Phi}(f)\mathbf{R}\mathbf{\Phi}(f)\mathbf{q}_j|^2)\end{aligned} \tag{12}$$

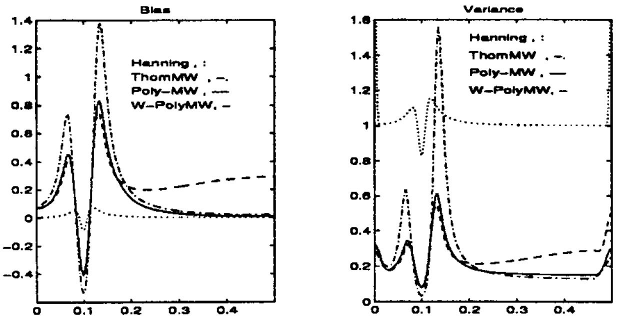

Figure 6: Theoretical bias and variance for Example 1.

according to Walden et al, [3]. The bias and variance calculated from these formulas are shown for the three process examples in section 4.

Example 1

The ThomMW spectrum estimates of the second order AR-process show the characteristic bias that broadens the peak, Figure 6. The PolyMW and W-PolyMW method have reduced the bias but still the single window method has as expected the lowest bias. However, the variance of the single window method is about 8 times larger than the multiple window methods. It can be seen that around the peak the PolyMW and W-PolyMW has smaller variation compared to the ThomMW method. Still the variance for the W-PolyMW method are comparable to the ThomMW method in the smooth spectrum region, f>0.2.

Example 2

For the third order Butterworth lowpass spectrum, the bias shows a breakdown behaviour at different frequencies, Figure 7. For the different methods the breakdown is around f=0.2, f=0.3 and f=0.4 for the PolyMW, ThomMW and W-PolyMW methods respectively. However, the variance shows results which are better than for the single window. The robustness of the PolyMW and W-PolyMW method are shown, since the variances are about the same size as the ThomMW method for f<0.1. This indicates that it is not that sensitive if one uses a method designed for peak estimation when the spectrum is smooth.

Example 3

For the 20th order Butterworth lowpass spectrum we take a closer look at the passband, f<0.1. The bias is slightly better for the ThomMW method than for the W-PolyMW method and so is the variance, Figure 8. However, one should remember that this is the spectrum type that the ThomMW method is designed for. The bias from the single window (Hanning) is better due to the better resolution and the variance is as expected about 8 times larger than the Multiple Window methods.

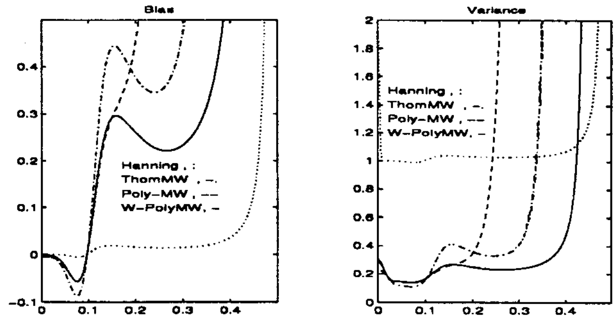

Figure 7: Theoretical bias and variance for Example 2.

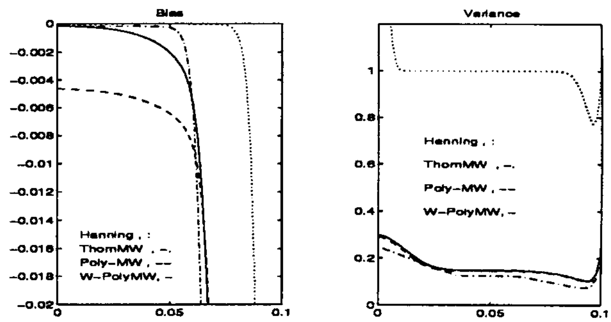

Figure 8: Theoretical bias and variance for Example 3.

6. CONCLUSIONS

This paper presents a non-parametric way of estimating the power spectrum with the use of Multiple Windows. These Multiple Windows are found as eigenvectors of the correlation matrix which have a bandlimited low-frequency dominant spectrum. The spectrum has a similar shape to the peaked spectrum of the stochastic process to be estimated. Having this a priori information, the eigenvectors can be better adapted to the signal. As these eigenvectors can have large sidelobes a penalty function is used to reduce the leakage from other frequencies.

The result shows that the proposed method (W-PolyMW) has smaller bias and variance than the Thomson Multiple Window method (ThomMW) for a peaked spectrum and still has robust behaviour for a smooth spectrum.

7. REFERENCES

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