LINEAR PARAMETRIC MODELS FOR SIGNALS WITH LONG-RANGE DEPENDENCE AND INFINITE VARIANCE

Stephen M. Kogon

Digital Signal Processing Laboratory
School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0250

Dimitris G. Manolakis

Institute for Space Research
Boston College
885 Centre Street
Chestnut Hill, MA 02159-1164

ABSTRACT

In this paper, two models of long-range dependence with finite and infinite variance that have recently been proposed in the mathematics literature are considered. The models are used for the characterization of experimental data in order to determine the possible benefits they offer over existing models. They are presented under a unified framework and their similarites and differences are investigated by applying the models to real world data in the form of infrared background signals.

1. INTRODUCTION

Long-range dependence is the presence of a significant correlation between observations of a signal separated by large time spans. It is closely linked with self-similar stochastic processes and random fractals which have been considered extensively, though only recently for signal processing applications. The phenomenon of long-range dependence, also referred to as persistence, can be found in many naturally occuring processes in diverse fields such as hydrology, geophysics, biological systems, and remote sensing. Signals with strong long-range correlations can not be modelled with traditional models, such as the autoregressive moving average (ARMA) model [1], since these models assume a rapidly decaying dependence that assumes distant observations to be uncorrelated. Instead, models specifically designed to characterize this long-range dependence must be used in order to optimize the performance of signal processing algorithms. Models of long-range dependence have previously been proposed and have found common use. Fractional Brownian motion (fBm) and its increment process fractional Gaussian noise (fGn) [2] have been considered extensively in the random fractals literature. The fractional ARIMA model has already gained popularity in the hydrology field [3] and has only recently been considered in the signal processing community [4].

These models, although good models of long-range dependence, are limited in their scope, since they make the assumption the data arises from a Gaussian process, which often is not true, as in the case of backgrounds of infrared scenes [5]. This type of data many times has impulsive behavior and thus high variability, an indication of a long-tailed distribution, which is certainly non-Gaussian. These probability distributions have as the name implies long "fat" tails and therefore infinite variance. Infinite variance means the second-order moments do not exist in theory (are infinite) and the data contains extreme values or "spikes" in practice. Stable distributions have been found to be good models of infinite variance processes [6]. By generalizing the distribution of the data in these existing models from Gaussian to α-stable, a new much richer class of signals can be characterized. Since the Gaussian distribution is a member of the family of stable distributions, the new models will contain the old models as special (finite variance) cases.

2. LONG-RANGE DEPENDENCE

As the name implies, long-range dependence is characterized by strong dependences at large lags and has led to such processes being referred to as long memory. The correlation at these large lags although small, by no means can be considered negligible. The dependence decays hyperbolically, rather than exponentially as in the case of the traditional short-memory models. Hyperbolic decay of the dependence implies a self-similar stochastic structure within the data, which refers to the fact the signal retains the same appearance at different levels of resolution. This can be formulated as

$$x(at) \stackrel{d}{=} a^H x(t) \tag{1}$$

for $a > 0$, where $\stackrel{d}{=}$ denotes equality in distribution and $H \in [0, 1]$ is the self-similarity or Hurst parameter. The concept of self-similarity is one of the cornerstones of the theory of random fractals, such as fBm, and explains their popularity as models of long-range dependent signals. Heuristically, the self-similarity parameter H can be thought of as a measure of the irreg-

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ularity of the signal, where smaller values correspond to greater irregularity.

Another trait of long-memory processes is the $1/f$ type behavior in the power spectral density, which is a power law relationship, i.e. $S(\omega) \sim \frac{1}{\omega^a}$ where a is a constant. When the power spectral density is plotted on a logarithmic scale, a linear relationship results. This has resulted in long-range dependent signals often being referred to as $1/f$ noise.

3. STABLE DISTRIBUTIONS

The family of stable distributions have been found to be good models of infinite variance [6], where the tails and the skewness of the distribution are characterized by the stable characteristic exponent α and the symmetry parameter β respectively. Contained in this family are the Gaussian ($\alpha = 2$, $\beta = 0$) and the Cauchy ($\alpha = 1$, $\beta = 0$) distributions.

A stable distribution is defined as a distribution for which the linear combination of independent identically distributed (iid) random variables will preserve the original distribution within a linear tranformation. This is known as the generalized central limit theorem and is given by

$$x_1 + x_2 + \ldots + x_n \stackrel{d}{=} ax + b \qquad (2)$$

where a and b are constants, x_i are iid random variables, and $\stackrel{d}{=}$ denotes equality in distribution. Another attractive property of stable distributions is the fact the distribution will be preserved within any linear model.

Although stable distributions have only recently been introduced to the signal processing community, a great deal of theory has already been developed [6]. A reason they have not gained widespread acceptance is the lack of an analytic closed form expression for the probability density function. However, this problem can be overcome by looking at the characteristic function $\Phi(\omega)$ given by

$$\Phi(\omega) = \begin{cases} e^{i\delta\omega - |\sigma\omega|^{\alpha}\left(1 - i\beta\frac{\omega}{|\omega|}\tan\frac{\pi\alpha}{2}\right)} & \text{if } \alpha \neq 1, \\ e^{i\delta\omega - |\sigma\omega|\left(1 - i\beta\frac{2}{\pi}\frac{\omega}{|\omega|}\ln|\omega|\right)} & \text{if } \alpha = 1. \end{cases} \qquad (3)$$

The parameters of the stable distribution are the stable characteristic exponent α ($0 < \alpha \leq 2$), the symmetry parameter β ($-1 \leq \beta \leq 1$), the spread parameter σ, and the location parameter δ. The shape of the distribution is completely specified by α and β, whereas δ and σ simply perform a linear transformation. In many cases a symmetric α-stable (SαS) distribution can be assumed ($\beta = 0$), for which the characteristic function simplifies to

$$\Phi(\omega) = e^{i\delta\omega - |\sigma\omega|^{\alpha}}. \qquad (4)$$

The lower the value of the characteristic exponent α, the "fatter" the tails of the probability density function become and the more impulsive the data is with greater probability of extreme values.

4. MODELS

The two most common classes of models for long-range dependence fall into the categories of either self-similar stochastic processes or fractional ARIMA. Although they have many similarities, they also have many important differences. When discussing models one has to make the distinction between stationary and non-stationary models. In this section fractional Lévy stable noise (fLsn) and the fractional ARIMA model driven by α-stable innovations are considered as stationary models of long-range dependence processes with finite or infinite variance. Their non-stationary counterparts are briefly discussed within the context of the individual models.

4.1. Fractional Lévy Stable Noise

The fractional Lévy stable noise (fLsn) model is a generalization of the well known fractional Gaussian noise (fGn) process. It is the increment process of the non-stationary fractional Lévy stable motion (fLsm) model [5]. As discussed earlier in this paper, the capabilities of the fGn model are limited since it makes the assumption of a Gaussian distribution. However, if the distribution is generalized from Gaussian to α-stable, a much richer family of processes can be described. The causal representation of the fLsn model is formulated by the stochastic integral [5]

$$x(t) = \int_{-\infty}^{t} \left[(t + 1 - \xi)^{H - \frac{1}{\alpha}} - (t - \xi)^{H - \frac{1}{\alpha}}\right] M_{\alpha}(d\xi) \qquad (5)$$

where $M_{\alpha}(d\xi)$ is a white SαS noise measure. A heuristic interpretation of this stochastic integral is to view it as a weighted "sum" of white α-stable noise, which is in fact what the discrete version of the fLsn process is. Discrete fLsn is obtained by using a summation in place of the integral in Eqn. 5, which amounts to sampling the continuous time fLsn process. Note, there are some aliasing issues to be addressed in this formulation, since continuous-time fLsn has an infinite bandwidth, but this discussion is beyond the scope of this paper. Note, the dependence structure of fLsn has been built around the point $H = \frac{1}{\alpha}$, with positive long-range dependence for $H > \frac{1}{\alpha}$, negative long-range dependence for $H < \frac{1}{\alpha}$, and independence for $H = \frac{1}{\alpha}$. Fig. 1 shows several one-dimensional realizations of fLsn with different values of α, with the same self-similarity parameter ($H = 0.5$). The extreme values in the realizations increase in number and magnitude for lower values of α. What appear to be trends is actually the presence of

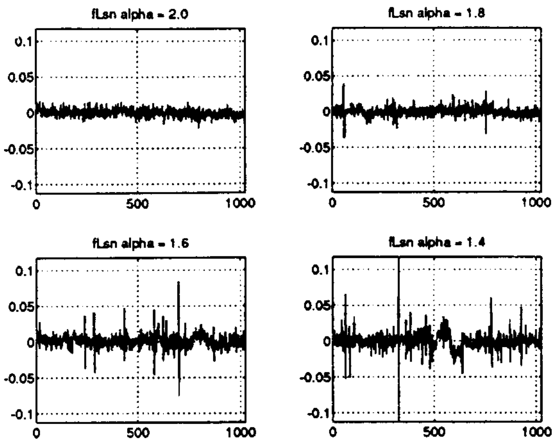

Figure 1: 1-D realizations of fLsn ($H = 0.7$).

long-range dependence. The non-stationary counterpart to fLsn is fractional Lévy stable motion (fLsm), which is formulated by modeling its increment process with fLsn. The fLsm process is then defined as the integration of the fLsn process.

The dependence structure of fLsn, although closely related to the special case of fGn, is somewhat difficult to formulate with existing techniques. Since fLsn is an α-stable process with infinite variance it does not make sense to talk about correlation functions, since the second-order moments of the process do not exist. Instead we must formulate the dependence in terms of the codifference function [6], which like the correlation function is zero in the case of independence and symmetric. In the Gaussian case ($\alpha = 2$), the codifference function reduces to the correlation function. Rather than go through the lengthy details of formulating the dependence structure of fLsn in terms of the codifference function, it is insightful to look at the correlation function of the fGn process, noting the dependence of the fLsn process is similar in nature. The correlation function of the discrete fGn process is

$$r[k] = \frac{\sigma^2}{2}\left\{\|k+1\|^{2H} - 2k^{2H} + \|k-1\|^{2H}\right\} \qquad (6)$$

and the power spectral density takes on the desired form, having a power law relationship ($S(\omega) \sim |\omega|^{1-2H}$) at frequencies near zero. It is quite evident the dependence is decaying hyperbolically, allowing this model to characterize the long-range correlations found in many natural events and not limiting it to Gaussian processes.

4.2. Fractional ARIMA

The fractional ARIMA model was first considered by Hosking [3] for the Gaussian case. The model is made up of two components: the ARMA and the fractional difference portions. The model is excited by white (iid) noise, which is passed through the two components of the model. Since the ARMA process is well understood this section will instead focus on the fractional difference portion of the model, but instead of limiting the inputs of the model to be Gaussian, they are generalized to be from a stable distribution. This like the fLsn model is capable of modeling long-range dependence processes with either finite or infinite variance.

The fractional differencing model is parameterized by d the difference parameter and is

$$x[n] = \sum_{k=0}^{\infty} h[n]\, w[n-k] \qquad (7)$$

where $w[n]$ is a sequence of iid α-stable random variables. The process is defined as the difference operation, with a transfer function of $H(z) = \frac{1}{(1-z^{-1})^d}$, where d is the difference parameter, taking on non-integer values. The impulse response $h[n]$ of this process is

$$h[n] = \frac{d(d-1)\cdots(d+n-1)}{n!} = \frac{(n+d-1)!}{k!\,(d-1)!}. \qquad (8)$$

The difference parameter is closely related to the self-similarity parameter H of a self-similar process (such as fLsn)

$$d = H - \frac{1}{\alpha}. \qquad (9)$$

Since the self-similarity parameter is $0 < H < 1$, the difference parameter is in the interval $-\frac{1}{\alpha} < d < H - \frac{1}{\alpha}$ (for stationary processes).

The long-range dependence of the fractional ARIMA process is somewhat difficult to characterize without the use of the codifference function [6], which requires a considerable amount of formulation in order to be properly understood. However, it is again insightful to consider the Gaussian case, for which the codifference and correlation functions are equivalent. A fractional ARIMA process driven by Gaussian innovations will have a correlation function of

$$r[k] = \frac{d(1+d)\cdots(k-1+d)}{(1-d)(2-d)\cdots(k-d)} \qquad (10)$$

which becomes proportional to k^{2d-1} as $k - \infty$. On the other hand the power spectral density is $S(\omega) = \left(2\sin\frac{1}{2}\omega\right)^{-2d}$ which becomes proportional to ω^{-2d} as $\omega \to 0$. Both the autocorrelation function and the power spectral density have the desired properties of a long-range dependence process and illustrate the type of behavior to expect for the infinite variance case.

5. RESULTS

The fLsn and fractional ARIMA model with α-stable innovations are applied to the modeling of signals from

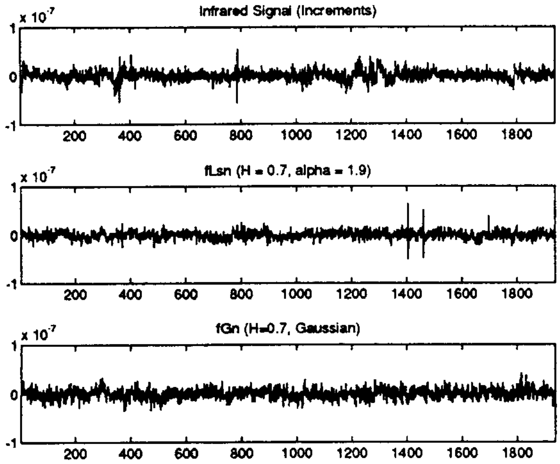

Figure 2: Fractional Lévy stable noise model results

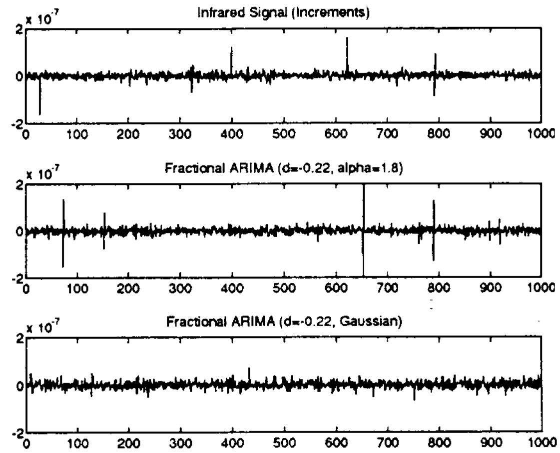

Figure 3: Fractional ARIMA results.

infrared (IR) scenes. The data is collected with airborne IR sensors while viewing clouds and terrain below. The data typically exhibit statistical self-similarity, so the self-similar process (fLsn) and fractional ARIMA models are natural choices. The data is also characterized by impulsive behavior. Many times in practice the resulting extreme values or "spikes" are considered to be outliers and are removed from the data. The two proposed models allow for the modeling of such behavior rather than ignoring it.

The IR data used are clearly non-stationary. However, the increments of properly selected short segments can be reasonably assumed stationary. The fLsn model characterizes the increment process and the resulting model is in essence fractional Lévy stable motion. The model is constructed by estimating the self-similarity parameter H and the stable characteristic exponent α [7]. The estimated parameters for the IR signal are $H = 0.83$ and $\alpha = 1.86$. The orginal IR signal and synthesized signals using the fLsn model and the fGn model are shown in Fig. 2. Note, the presence of extreme values in the fLsn synthesized signal, where the distribution of the data was used in the model. On the other hand the fGn synthesized signal does not capture the impulsive behavior, since a Gaussian distribution was assumed. Similar results are obtained when using the fractional ARIMA model with α-stable innovations. Again, the IR signal is non-stationary with stationary increments. The model is applied to the increments by estimating the parameters in the same fashion as for the fLsn model and exploiting the relation between d and H (Eqn. 9). The estimated parameters are $d = -0.22$ and $\alpha = 1.80$, which are used to synthesize the fractional ARIMA signals with α-stable and Gaussian innovations [8]. The resulting signals are shown in Fig. 3. As in the case of the fLsn model, if the distribution of the data is ignored and assumed to be Gaussian, the resulting model is an inadequate description of the underlying process.

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