PERFORMANCE ANALYSIS FOR A CLASS OF AMPLITUDE MODULATED POLYNOMIAL PHASE SIGNALS

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ABSTRACT

We consider the parameter estimation problem for a class of amplitude modulated polynomial phase signals (PPS), observed in noise. The main contributions of this paper are: (1) We prove that the High-order Ambiguity Function (HAF) is invariant to certain types of amplitude modulation; thus, phase parameter estimation proceeds as in the constant amplitude case. (2) We derive the Cramér-Rao bounds for both the amplitude and phase parameters, when the additive noise is white Gaussian. (3) We show that the HAF is almost additive for multi-component PPS. (4) We establish the covariance bounds for the nonlinear least squares estimator when the additive noise is (non)Gaussian, and satisfies some weak mixing conditions.

1. INTRODUCTION

Many real life signals are nonstationary, and some of them can be modeled as amplitude modulated (AM) and/or frequency modulated (FM) signals. Friedlander and Francos [2] considered the case where both the amplitude and phase are linear combinations of known basis functions; they presented the maximum likelihood (ML) estimator, and derived the corresponding Cramér-Rao bounds (CRB) for the additive white Gaussian noise case.

Polynomial phase signals (PPS) are obtained when the basis functions for the phase are $\{t^m\}_{m=0}^{M}$. Constant amplitude chirp signals ($M = 2$) are studied in [4] and [1], and constant amplitude PPS of general order M are investigated systematically by Peleg, Porat, and Friedlander (see e.g., [6, Ch. 12], and references therein). Results for (stationary) random amplitude PPS are reported in [7], [8].

In this paper, we consider multi-component PPS with deterministic but time-varying (TV) amplitudes,

$$s(t) = \sum_{l=1}^{L} \rho_l(t/T;\underline{\theta}_{\rho_l})\, e^{j\sum_{m_l=0}^{M_l} a_{l m_l} t^{m_l}} \qquad (1)$$

$t = 0, 1, \ldots, T-1$. The amplitude function $\rho_l(t/T;\underline{\theta}_{\rho_l})$ is parameterized by $\underline{\theta}_{\rho_l}$ and satisfies the following assumptions [3]: **(a1)** $\rho_l(u;\underline{\theta}_{\rho_l})$ is a real and continuous function of bounded variation for $u \in [0,1]$, and vanishes for $u \notin [0,1]$; **(a2)** $\rho_l(u;\underline{\theta}_{\rho_l})$ is differentiable in $\underline{\theta}_{\rho_l}$ and the derivative is of bounded variation in u. The class of functions satisfying **(a1)** and **(a2)** includes the constant amplitude model, the transient model, the linear decay model, the polynomial model and so on [3],[10],[11]. Applications of the deterministic AM PPS model include: seismic signal processing (damped multi-component chirps), processing of Doppler radar signals in a fading environment, and modeling of speech signals, to name only a few. Simulation results and details of proofs will be given in [9] and [12].

2. AMPLITUDE MODULATION AND HAF

Let us first consider the single component version of (1),

$$s(t) = \rho(t/T;\underline{\theta}_\rho)\, e^{j\sum_{m=0}^{M} a_m t^m} = \rho(t/T;\underline{\theta}_\rho)\, e^{j\phi(t;\underline{\theta}_\phi)}, \quad (2)$$

where $\underline{\theta}_\phi := [a_0\ a_1\ \ldots\ a_M]'$ is the phase parameter vector, and $\phi(t;\underline{\theta}_\phi) := \sum_{m=0}^{M} a_m t^m$. The high-order instantaneous moment (HIM) (see e.g., [6, Sec. 12.6]) is defined as

$$\mathcal{P}_M[s(t);\tau] := \prod_{q=0}^{M-1} [s^{(*q)}(t-q\tau)]^{\binom{M-1}{q}}, \qquad (3)$$

where $s^{(*q)}(t) := s(t)$ for q even, and $s^{(*q)}(t) := s^*(t)$ for q odd. The high-order ambiguity function (HAF), is defined as the Fourier Series (FS) of the HIM,

$$P_M[s;\alpha,\tau] := \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathcal{P}_M[s(t);\tau]\, e^{-j\alpha t}. \qquad (4)$$

It can be shown [10, 11] that substitution of (2) and (3) into (4) yields, under (a1)-(a2), for finite τ,

$$P_M[s;\alpha,\tau] = \left[\int_0^1 [\rho(u)]^{2^{M-1}} du\right] e^{j\tilde{\phi}}\, \delta(\alpha - \tilde{\omega}) \quad (5)$$

$$\tilde{\omega} := M!\, \tau^{M-1} a_M, \qquad (6)$$

where $\tilde{\phi}$ is a function of M, τ, a_{M-1}, and a_M [6, p. 395], and $\delta(\cdot)$ is the Kronecker delta function.

From (6), we conclude that: *Polynomial phase signals with time-varying amplitudes satisfying* (**a1**) *and* (**a2**), *have the same HAF, to within a constant scale factor, as the corresponding constant amplitude PPS.*

In practice, we observe a noisy version of (1),

$$x(t) = s(t) + g(t) = \rho(t/T;\underline{\theta}_\rho)\, e^{j\sum_{m=0}^{M} a_m t^m} + g(t), \quad (7)$$

where it is assumed that: **(a3)** $g(t)$ is zero-mean, white circular complex Gaussian with finite variance σ_g^2.

A natural estimate of (3) is $\hat{\mathcal{P}}_M[s(t);\tau] := \mathcal{P}_M[x(t);\tau]$, where the latter is defined similar to (3). The HAF estimator is given by

$$\hat{P}_M[s;\alpha,\tau] := \frac{1}{T}\sum_{t=0}^{T-1} \hat{\mathcal{P}}_M[s(t);\tau]\, e^{-j\alpha t}, \qquad (8)$$

and can be efficiently computed via the FFT. It is proved in [10, 11] that $\hat{P}_M[s;\alpha,\tau]$ is an asymptotically unbiased and mean square sense consistent estimator of $P_M[s;\alpha,\tau]$.

Based on (6), one can estimate a_M from the peak location of $|\hat{P}_M[s;\alpha,\tau]|$, multiply $x(t)$ by $\exp(-j\hat{a}_M t^M)$, and

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repeat the procedure to obtain a_{M-1}, and so on. Instead of computing the FT of the HIM estimate, we can also apply high resolution algorithms - such as the Kumaresan-Tufts, matrix pencil, and MUSIC algorithms - to the HIM estimate. Their performance and relative merits are discussed in [10],[11], where special cases of $\rho(t;\cdot)$ are also considered.

3. CRAMÉR-RAO BOUNDS

The log-likelihood function for $x(t)$ in (7) is given by

$$\Lambda = -\frac{1}{\sigma_g^2}\sum_{t=0}^{T-1}\left|x(t) - \rho(t/T;\underline{\theta}_\rho)\, e^{j\phi(t;\underline{\theta}_\phi)}\right|^2 \quad (9)$$

If we denote the kth-element of $\underline{\theta}_\rho$ by θ_{ρ_k}, and the lth-element of $\underline{\theta}_\phi$ by θ_{ϕ_l}, then the entry of the Fisher information matrix (FIM) corresponding to parameters θ_{ρ_k} and θ_{ϕ_l} is $J_{\theta_{\rho_k},\theta_{\phi_l}} := -E[\partial^2\Lambda\,/\partial\theta_{\rho_k}\,\partial\theta_{\phi_l}]$. It is not difficult to show that $J_{\theta_{\rho_k},\theta_{\phi_l}} = 0$, and hence the FIM for the amplitude part is decoupled from that for the phase part. The (k,l) entry of the FIM for $\underline{\theta}_\rho$ is

$$J_{\rho,kl} = \frac{2}{\sigma_g^2}\sum_{t=0}^{T-1}\frac{\partial\rho(t/T;\underline{\theta}_\rho)}{\partial\theta_{\rho_k}}\,\frac{\partial\rho(t/T;\underline{\theta}_\rho)}{\partial\theta_{\rho_l}} \quad (10)$$

$$\rightarrow \frac{2T}{\sigma_g^2}\int_0^1\frac{\partial\rho(u;\underline{\theta}_\rho)}{\partial\theta_{\rho_k}}\frac{\partial\rho(u;\underline{\theta}_\rho)}{\partial\theta_{\rho_l}}\,du. \quad (11)$$

From (10), we see that $J_{\rho,kl} = O(T)$, $\forall k,l$, and hence $\mathrm{CRB}(\hat{\theta}_{\rho_k}) = O(T^{-1})$. Moreover, $J_{\rho,kl}$ does not involve $\phi(t)$, and therefore $\mathrm{CRB}(\hat{\theta}_{\rho_k})$ is not a function of $\underline{\theta}_\phi$.

For the exponentially damped harmonic, $\rho(t/T) = \rho_o \exp(-bt/T)$, $\underline{\theta}_\rho := [\rho_o,\ b]'$, and from (10) we have

$$\mathbf{J}_\rho = \frac{2\rho_o^2 T}{\sigma_g^2}\begin{bmatrix}\frac{\epsilon_0}{\rho_o^2} & -\frac{\epsilon_1}{\rho_o}\\ -\frac{\epsilon_1}{\rho_o} & \epsilon_2\end{bmatrix}, \quad (12)$$

$$\epsilon_k := T^{-1}\sum_{t=0}^{T-1}(t/T)^k\exp(-2bt/T),\ k\geq 0. \quad (13)$$

The diagonal elements of $\mathbf{J}_\rho^{-1}$ yield the corresponding CRBs,

$$\mathrm{CRB}(\hat{\rho_o}) = \frac{\sigma_g^2}{2T}\,\frac{\epsilon_2}{\epsilon_0\epsilon_2-\epsilon_1^2}, \quad (14)$$

$$\mathrm{CRB}(\hat{b}) = \frac{\sigma_g^2}{2\rho_o^2 T}\,\frac{\epsilon_0}{\epsilon_0\epsilon_2-\epsilon_1^2}. \quad (15)$$

Note that the results of (14) and (15) hold for *any* $\phi(t)$, not just polynomials. For the special case of damped harmonics, these results reduce to those in [3]. Note that since each ϵ_k is $O(1)$, the CRBs in (14) and (15) are both $O(T^{-1})$.

Now for the phase parameters, we have

$$J_{\phi,kl} = \frac{2}{\sigma_g^2}\sum_{t=0}^{T-1}\rho^2(t/T)\,\frac{\partial\phi(t)}{\partial\theta_{\phi_k}}\,\frac{\partial\phi(t)}{\partial\theta_{\phi_l}}. \quad (16)$$

Unlike $J_{\rho,kl}$, $J_{\phi,kl}$ in (16) can be $O(T^m)$ for different m's depending on the specific choice of $\phi(t)$. For the polynomial phase function $\phi(t) = \sum_{m=0}^{M} a_m t^m$, which is of interest in this paper, $\theta_{\phi_k} := a_k$, and $\partial\phi(t)/\partial a_k = t^k$. From (16), we conclude that for $k,l = 0,1,\ldots,M$,

$$J_{\phi,kl} = \frac{2T^{k+l+1}}{\sigma_g^2}\,\frac{1}{T}\sum_{t=0}^{T-1}(t/T)^{k+l}\,\rho^2(t/T) \quad (17)$$

$$\rightarrow \frac{2T^{k+l+1}}{\sigma_g^2}\int_0^1 u^{k+l}\rho^2(u;\underline{\theta}_\rho)\,du; \quad (18)$$

hence, $J_{\phi,kl} = O(T^{k+l+1})$. The inverse of the $(M+1)\times(M+1)$ FIM $\mathbf{J}_\phi$, whose (k,l) entry is given by (17), therefore yields $\mathrm{CRB}(\hat{a}_m) = O(T^{-2m-1})$ as diagonal elements.

Exponentially damped chirp ($M = 2$) processes are of particular interest in applications such as Vibroseis. For such a process, eq. (17) yields for $k,l = 0,1,2$,

$$\mathbf{J}_\phi = \frac{2\rho_o^2 T}{\sigma_g^2}\begin{bmatrix}\epsilon_0 & T\,\epsilon_1 & T^2\,\epsilon_2\\ T\,\epsilon_1 & T^2\,\epsilon_2 & T^3\,\epsilon_3\\ T^2\,\epsilon_2 & T^3\,\epsilon_3 & T^4\,\epsilon_4\end{bmatrix}, \quad (19)$$

where ϵ_k is given by (13). The diagonal elements of $\mathbf{J}_\phi^{-1}$ are

$$\mathrm{CRB}(\hat{a}_0) = \frac{\sigma_g^2}{2\rho_o^2}\,\frac{\epsilon_2\epsilon_4-\epsilon_3^2}{TD}, \quad (20)$$

$$\mathrm{CRB}(\hat{a}_1) = \frac{\sigma_g^2}{2\rho_o^2}\,\frac{\epsilon_0\epsilon_4-\epsilon_2^2}{T^3 D}, \quad (21)$$

$$\mathrm{CRB}(\hat{a}_2) = \frac{\sigma_g^2}{2\rho_o^2}\,\frac{\epsilon_0\epsilon_2-\epsilon_1^2}{T^5 D}, \quad (22)$$

where $D := \epsilon_0\epsilon_2\epsilon_4 - \epsilon_1^2\epsilon_4 - \epsilon_0\epsilon_3^2 + 2\epsilon_1\epsilon_2\epsilon_3 - \epsilon_2^3$. It follows that the CRBs in (20), (21), and (22) are $O(T^{-1})$, $O(T^{-3})$, and $O(T^{-5})$. Moreover, the CRBs for the constant amplitude chirp model ($b = 0$) can be obtained from (20)-(22) with $\epsilon_k \approx T/(k+1)$, and are given by $\mathrm{CRB}(\hat{a}_0) \approx 4.5\sigma_g^2/(T\rho_o^2)$, $\mathrm{CRB}(\hat{a}_1) \approx 96\sigma_g^2/(T^3\rho_o^2)$, $\mathrm{CRB}(\hat{a}_2) \approx 90\sigma_g^2/(T^5\rho_o^2)$. These results agree with those in [1].

4. MULTI-COMPONENT PPS AND HAF

The HIM is a nonlinear operator, hence it is expected that cross terms appear when one computes the HIM of a multi-component process. In general, $\mathcal{P}_M$ of an $L-$component signal introduces as many as $L^{2^{M-1}} - L$ cross terms, which is 2 for $L = M = 2$, and is 14 for $L = 2, M = 3$. The objective of this section is to argue that the cross terms almost always disappear in the HAF domain; i.e., after the limiting FS operation. This implies that the HAF of a multi-component PPS can almost always be approximated by the sum of HAFs of individual PPS components.

For simplicity, we discuss here the two component ($L = 2$) case and constant amplitude PPS. Results for general multi-component and/or TV amplitude PPS follow similarly. We start with chirp signals ($M = 2$), which are modeled in discrete-time as,

$$s(t) = \rho_1\,e^{j(a_{12}t^2+a_{11}t+a_{10})} + \rho_2\,e^{j(a_{22}t^2+a_{21}t+a_{20})}. \quad (23)$$

We assume w.l.o.g. that ρ_1, ρ_2 are real, and $\rho_1 > \rho_2 > 0$. The instantaneous 2nd-order moment of (23) is given by:

$$\mathcal{P}_2[s(t);1] = \rho_1^2\,e^{2ja_{12}t}\,e^{j(a_{11}-a_{12})} + \rho_2^2\,e^{2ja_{22}t}\,e^{j(a_{21}-a_{22})} \quad (24)$$

$$+ 2\rho_1\rho_2\,e^{j(a_{12}-a_{22})t^2+j(a_{11}-a_{21}+2a_{22})t+j(a_{21}-a_{22}+a_{10}-a_{20})} \quad (25)$$

$$+ 2\rho_1\rho_2\,e^{j(a_{22}-a_{12})t^2+j(a_{21}-a_{11}+2a_{12})t+j(a_{11}-a_{12}+a_{20}-a_{10})}. \quad (26)$$

The "auto" terms in (24) are the 2nd-order HIM of the individual components and produce spectral lines at $\alpha = 2a_{12}$ and $2a_{22}$ with magnitudes ρ_1^2 and ρ_2^2 respectively. We are interested in evaluating the contributions of the cross terms (25) and (26) to the HAF, $P_2[s;\alpha,1]$. Such contributions have been characterized as non-random noise.

Since a factor $\exp(j\omega_0 t)$ only shifts spectra lines in the FS domain, and $\exp(j\phi_0)$ has no effect on the magnitude, we shall focus on the behavior of $\mathrm{FS}[\exp(j\nu_2 t^2)]$ only, where $\nu_2 := a_{12} - a_{22}$.

Surprisingly, although $\exp(j\nu_2 t^2)$ is aperiodic in continuous time $\forall\ \nu_2$, it is periodic in discrete-time for $\nu_2 = 2\pi\mathcal{N}/\mathcal{D}$, where $\mathcal{N}, \mathcal{D}$ are co-prime integers. To see this, recall that any integer t can be written as $t = i\mathcal{D} + k$, where $i = [t/\mathcal{D}]$, and $k \in [0, \mathcal{D}-1]$. It follows easily that

$$e^{j\nu_2 t^2} = e^{j2\pi(k^2+2i\mathcal{D}k+i^2\mathcal{D}^2)\mathcal{N}/\mathcal{D}} = e^{j2\pi k^2\mathcal{N}/\mathcal{D}}, \qquad (27)$$

which is periodic. It turns out [12] that when $\mathcal{D}$ is a multiple of 4, then $\mathcal{D}/2$ is the period, otherwise $\mathcal{D}$ is the period. Since $\exp(j2\pi t^2\mathcal{N}/\mathcal{D})$ is periodic, its FS coefficient, denoted as $h(\alpha)$, contains spectral lines. We can show that when $\mathcal{D}$ is even, $h(\alpha)$ consists of $\mathcal{D}/2$ lines, and $\max_\alpha |h(\alpha)| = \sqrt{2/\mathcal{D}}$; whereas when $\mathcal{D}$ is odd, $h(\alpha)$ consists of $\mathcal{D}$ lines, and $\max_\alpha |h(\alpha)| = \sqrt{1/\mathcal{D}}$.

Because line spectra are produced only when ν_2 is of the form 2π times a rational, and rational numbers have measure zero, we conclude that lines almost never occur in $h(\alpha)$. One may argue that since any real number can be approximated arbitrarily closely by a rational number, line spectra should be seen frequently. Our answer is that $\mathcal{D}$ has to be sufficiently large to obtain a good approximation, and as $\mathcal{D} \to \infty$, the peak strength $\sqrt{2/\mathcal{D}}$ or $\sqrt{1/\mathcal{D}}$ goes to zero, and hence there will be no lines. If significant lines do show up, ν_2 must be of the form $2\pi\mathcal{N}/\mathcal{D}$ with $\mathcal{D}$ small. Therefore we assert that in general cross terms do not confuse the spectral lines which are due to individual PPS components, and the HAF is essentially additive.

From (24)-(26), if ρ_1^2 is always larger than $2\rho_1\rho_2$ times the maximum of $|h(\alpha)|$, due to the cross terms, then one can always correctly identify the largest signal component, estimate its parameters, remove this component, and reduce the number of components by one [5]. The condition stated in [5] for this to be feasible is $\rho_1/\rho_2 > 2$. We show next that one can significantly weaken this constraint.

First, with $\tau = 1$, the leading chirp coefficients must satisfy $|a_{12}| < \pi/2$ and $|a_{22}| < \pi/2$ in order to satisfy the HAF-based identifiability conditions. This implies that $|\nu_2| = |a_{12} - a_{22}| < \pi$, and hence $\mathcal{N}/\mathcal{D} < 1/2$. Assume $\rho_1/\rho_2 > 1$; next, we identify the worst case scenarios which put additional constraints on ρ_1/ρ_2: **(c1)** $\mathcal{D} = 4$, $\mathcal{N} = 1$, $|a_{12} - a_{22}| = \pi/2$, which requires $\rho_1/\rho_2 > \sqrt{2}$, and **(c2)** $\mathcal{D} = 3$, $\mathcal{N} = 1$, $|a_{12} - a_{22}| = 2\pi/3$, which requires $\rho_1/\rho_2 > 2/\sqrt{3}$. Therefore we conclude that if $|a_{12} - a_{22}| \neq \pi/2$ or $2\pi/3$, then the successive estimation algorithm described in [5] can be implemented, for *any* $\rho_1/\rho_2 > 1$. Otherwise, one needs to ensure $\rho_1/\rho_2 > \sqrt{2}$ or $2/\sqrt{3}$. This is a much weaker condition than the one stated in [5].

For a general Mth-order PPS, we prove similarly that $\exp(j\nu_M t^M)$ is periodic if $\nu_M = 2\pi\mathcal{N}/\mathcal{D}$. For M prime and $\mathcal{D}$ an integer multiple of M^2, its period is $\mathcal{D}/M$; otherwise, the period is $\mathcal{D}$. The situation where M is a composite number will be discussed in [12].

For Mth-order PPS, cross terms in the HIM are of the general form $\prod_{m=0}^{M} \exp(j\nu_m t^m)$, and is periodic only when *every* ν_m is of the form $2\pi\mathcal{N}/\mathcal{D}$, which is rather unlikely. Similar arguments can be used to conclude that cross terms do not contribute much to the HAF. Detailed analysis will be provided in [12].

5. NON-LINEAR LEAST SQUARES ESTIMATOR

Although the FFT-based HAF method is easy to implement, it is suboptimal; hence, we consider the non-linear least squares estimator (NLLSE), for both the phase and amplitude parameters. The main result is given by Theorem 1, and is a generalization of results by Hasan [3], who considers the pure harmonic, i.e., $(M = 1)$ in (7).

Consider the noisy monocomponent model in (7). In addition to **(a1)** and **(a2)**, conditions **(a3)** and **(a4)** below are also assumed to be in force.

(a3). The noise sequence $g(t)$ is strictly stationary, circularly symmetric (i.e., its real and imaginary parts have the same distribution, and are independent of each other) and purely non-deterministic, with zero mean, $E|g(t)|^k < \infty$, $k = 0, 1, \ldots$, and satisfies the mixing condition, $\sum_\tau |c_{kg}(\underline{\tau})| < \infty$, $k = 2, 3, \cdots$.

(a4). The basic identifiability assumption is:

$$(\underline{\theta}_\rho = \underline{\theta}'_\rho) \ \text{ iff } \int_0^1 |\rho(u;\underline{\theta}_\rho) - \rho(u;\underline{\theta}'_\rho)|^2\, du = 0 \qquad (28)$$

Let $\underline{\theta}_\phi := [a_0, a_1 T, \ldots, a_M T^M]'$, $\underline{\theta} := [\underline{\theta}_\rho, \underline{\theta}_\phi]$, and let $\underline{\theta}_o$ denote the true parameters. Note that we have re-defined $\underline{\theta}_\phi$. For convenience, we do not explicitly denote the dependence of $\underline{\theta}_\phi$, $\underline{\theta}$, $\underline{\theta}_o$ and $s(\cdot)$ upon T. The NLLSE, $\hat{\underline{\theta}}$, minimizes

$$Q_T(\underline{\theta}) \quad := \quad \sum_{t=0}^{T-1} |x(t) - s(t;\underline{\theta})|^2 \qquad (29)$$

and can be initialized by estimates obtained from the sub-optimal HAF scheme.

5.1. Rate of convergence & Consistency

Lemma 1 *Let $\hat{\underline{\theta}}$ denote an estimate of $\underline{\theta}$ based on T samples. Then,*

$$J = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} |s(t;\hat{\underline{\theta}}) - s(t;\underline{\theta})|^2 = 0 \qquad (30)$$

only if $\hat{\underline{\theta}}_\rho = \underline{\theta}_\rho + o_p(1)$, and $\hat{a}_k = a_k + o_p(T^{-k})$. □

Proof. Omitted due to lack of space; see [9].

Note that the rate of convergence of the k-th phase parameter is of order $1/T^k$. Consistency of the NLLSE follows directly from Lemma 1 and assumption **(a3)**.

5.2. Covariance Expressions

Since the NLLSE is consistent, the Taylor expansion

$$s(t;\hat{\underline{\theta}}) = s(t;\underline{\theta}_o) + (\hat{\underline{\theta}} - \underline{\theta}_o)\ \nabla_{\underline{\theta}} s(t;\underline{\theta})|_{\underline{\theta}=\underline{\theta}_o} + o_p(1)\ , \quad (31)$$

holds for large T; the order $o_p(1)$ for the remainder term follows from Lemma 1 (recall the rates of convergence, and note the scaling). We substitute (31) and (7) in (29), to obtain a quadratic function in $\tilde{\underline{\theta}} := \hat{\underline{\theta}} - \underline{\theta}_o$; the least squares solution is given by the linear system of equations,

$$\tilde{\underline{\theta}} = \mathbf{A}_T^{-1}\mathbf{b}_T \qquad (32)$$

where

$$A_T(m,n) \quad := \quad 2\ \text{Real} \sum_{t=0}^{T-1} \frac{\partial s(t;\underline{\theta})}{\partial\theta_m}\frac{\partial s^*(t;\underline{\theta})}{\partial\theta_n}$$

$$b_T(m) \quad := \quad 2\ \text{Real} \sum_{t=0}^{T-1} \frac{\partial s(t;\underline{\theta})}{\partial\theta_m} g^*(t)$$

Vector $\mathbf{b}_T$ has zero mean; hence $\tilde{\underline{\theta}}$ has zero mean, and since matrix $\mathbf{A}_T$ is non-random (and real, symmetric), we have

$$\text{cov}\ (\tilde{\underline{\theta}}) \ = \ E\{\tilde{\underline{\theta}}\tilde{\underline{\theta}}^H\} \ = \ \mathbf{A}_T^{-1}\ \text{cov}\ (\mathbf{b}_T)\ \mathbf{A}_T^{-1}\ . \qquad (33)$$

We need to evaluate the asymptotic values of the component terms. We will find it convenient to define the following:

$$\int_0^1 u^{m+n}\rho(u;\underline{\theta}_\rho)\rho(u;\underline{\theta}_\rho)\,du \quad := \quad \gamma_{m,n}(\underline{\theta}_\rho) \qquad (34)$$

$$\int_0^1 \frac{\partial\rho(u;\underline{\theta}_\rho)}{\partial\theta_{\rho,m}}\frac{\partial\rho(u;\underline{\theta}_\rho)}{\partial\theta_{\rho,n}}\,du \quad := \quad \tilde{\gamma}_{m,n}(\underline{\theta}_\rho) \qquad (35)$$

$$\int_0^1 \left|\frac{\partial\rho(u;\underline{\theta}_\rho)}{\partial\theta_{\rho,m}}\frac{\partial\rho(u;\underline{\theta}_\rho)}{\partial\theta_{\rho,n}}\right|\,du \quad := \quad \vartheta_{m,n}(\underline{\theta}_\rho) \qquad (36)$$

$$\int_0^1 \left|\frac{\partial\rho(u;\underline{\theta}_\rho)}{\partial\theta_{\rho,m}}\right|\rho(u;\underline{\theta}_\rho)u^n\,du \quad := \quad \tilde{\vartheta}_{m,n}(\underline{\theta}_\rho)\,. \qquad (37)$$

By assumptions **(a1)-(a2)**, the elements of the matrix $T^{-1}\mathbf{A}_T$ are given asymptotically by,

$$\mathbf{H} := \lim_{T\to\infty}\frac{1}{T}\mathbf{A}_T = 2\begin{bmatrix} \{\tilde{\gamma}_{k,\ell}(\underline{\theta}_\rho)\} & \mathbf{0} \\ \mathbf{0} & \{\gamma_{m,n}(\underline{\theta}_\rho)\}\end{bmatrix} \qquad (38)$$

where $\tilde{\gamma}_{k,\ell}$ and $\gamma_{m,n}$ are defined in (34) and (35); k,ℓ range from 1 to N_ρ, the number of amplitude-related parameters, and m,n range from 0 to M. Notice that matrix $\mathbf{H}$ is block-diagonal: the amplitude and phase parameters are uncoupled.

The (m,n) element of the covariance matrix, $\mathbf{C}_{b,T}$, of the zero-mean $\mathbf{b}_T$ is given by

$$C_{b,T}(m,n) = 2\ \mathrm{Re}\sum_{t=0}^{T-1}\sum_{u=0}^{T-1}R_g(t-u)\frac{\partial s^*(t;\underline{\theta})}{\partial\theta_m}\frac{\partial s(u;\underline{\theta})}{\partial\theta_n} \quad (39)$$

where $R_g(\tau) = E\{g^*(t)g(t+\tau)\}$; two terms have dropped out since circular symmetry implies $E\{g(t)g(t+\tau)\} = 0$.

For the amplitude parameters, we have to evaluate

$$\beta_T \quad := \quad \sum_{t=-T+1}^{T-1}\sum_{u=0}^{T-1}R_g(t)\frac{\partial\rho(\frac{t+u}{T};\underline{\theta}_\rho)}{\partial\theta_{\rho,m}}\frac{\partial\rho(\frac{u}{T};\underline{\theta}_\rho)}{\partial\theta_{\rho,n}} \times e^{j\phi(u;\underline{\theta}_\phi)-j\phi(t+u;\underline{\theta}_\phi)}\,.$$

For $M > 2$, it is generally hard to evaluate β_T, since $\exp(j\phi(t))$ is not of bounded variations. Let

$$\bar{\sigma}_g^2 := \sum_{\tau=-\infty}^{\infty}|R_g(\tau)| < \infty\,, \qquad (40)$$

where the inequality follows by Assumption **(a3)**. Now,

$$|\beta_T| \leq \sum_{t=0}^{T-1}\sum_{u=0}^{T-1}|R_g(t)|\left|\frac{\partial\rho(\frac{t+u}{T};\underline{\theta}_\rho)}{\partial\theta_{\rho,m}}\frac{\partial\rho(\frac{u}{T};\underline{\theta}_\rho)}{\partial\theta_{\rho,n}}\right| \approx T\bar{\sigma}_g^2\,\vartheta_{m,n}(\underline{\theta}_\rho)\,.$$

Similarly, for the phase parameters, we obtain the upper bound $T\bar{\sigma}_g^2\,\gamma_{m,n}(\underline{\theta}_\rho)$. For the cross-parameters, θ_{ρ_k} and θ_{ϕ_l}, we obtain the bound $T\bar{\sigma}_g^2\tilde{\vartheta}_{k,l}(\underline{\theta}_\rho)$.

If $g(t)$ is white, $R_g(t-u) = \sigma_g^2\delta(t-u)$; the double summation in (39) collapses to a single summation over $t = u$, and it is easy to show that the amplitude and phase parameters are decoupled. Using (34) and (35), we obtain $\mathbf{C}_{b,T} = T\sigma_g^2\mathbf{H}$, and

$$\mathrm{cov}\ (\hat{\underline{\theta}}) \approx \frac{\sigma_g^2}{T}\mathbf{H}^{-1}\,. \qquad (41)$$

These results reduce to those of Peleg-Porat who consider the special case of a Gaussian $g(t)$, and $\rho(u;\underline{\theta}_\rho) \equiv \rho_o$. Note that the matrix $\mathbf{H}$ is a Hilbert matrix in the constant amplitude case; we can obtain approximations for moderate sample sizes by retaining terms of order $O(1)$ in $\mathbf{H}$. Our results are summarized in Theorem 1.

Theorem 1 *Under assumptions* **(a1)-(a4)**, *the NLLSE is asymptotically normal and unbiased. If $g(t)$ is white, the covariance matrix is given by*

$$\mathrm{cov}\ (\tilde{\underline{\theta}}) \approx \frac{\sigma_g^2}{T}\mathbf{H}^{-1} \qquad (42)$$

where matrix $\mathbf{H}$ *is defined in (38). In the case of colored noise, we obtain an element-wise upper bound*

$$\mathrm{cov}\ (\tilde{\underline{\theta}}) \ \circ\!\leq\ \frac{\bar{\sigma}_g^2}{T}\,|\mathbf{H}^{-1}|\ \mathbf{H_c}\ |\mathbf{H}^{-1}| \qquad (43)$$

where $|A|$ denotes the absolute value and $\circ\leq$ denotes element-wise inequality, and

$$\mathbf{H}_c := 2\begin{bmatrix} \{\vartheta_{k,\ell}(\underline{\theta}_\rho)\} & \{\tilde{\vartheta}_{k,n}(\underline{\theta}_\rho)\} \\ \{\tilde{\vartheta}_{\ell,m}(\underline{\theta}_\rho)\} & \{\gamma_{m,n}(\underline{\theta}_\rho)\}\end{bmatrix}\,; \qquad (44)$$

where γ, $\tilde{\gamma}$, ϑ and $\tilde{\vartheta}$ are defined in (34)-(37), and $\bar{\sigma}_g^2$ in (40). The elements of matrices $\mathbf{H}$ *and* $\mathbf{H}_c$ *are evaluated at the true parameter vector $\underline{\theta}_o$.* □

Based on Theorem 1, we assert that each parameter in $\underline{\theta}_\rho$ has variance $O(T^{-1})$, whereas $\mathrm{var}(\hat{a}_m) = O(T^{-2m-1})$. Asymptotic normality follows along the lines of Hasan [3] under assumption **(a3)**. Under the stronger assumption of absolute summability of cumulants of $g(t)$, asymptotic normality follows immediately from the ergodicity results of Dandawate-Giannakis (by assumptions **(a1)-(a2)**, the partials of $s(\cdot)$ are bounded).

The multi-component case is discussed in [9]. It will be seen that the amplitudes and phase parameter estimates are generally not decoupled; and that phase (amplitude) estimates of different components are also not decoupled. Note that even in the case of white noise, the FIM (CRB) is block diagonal only if $L = 1$ (single component) or $M = 1$ (pure harmonic). Here the periodicity issues discussed in Section 4. are important.

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