BET, THE BICORSPECTRAL TRANSFORM
DEFINITION, PROPERTIES, IMPLEMENTATION & APPLICATIONS

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ABSTRACT

This paper introduces a new time-frequency representation called the bicorspectral transform BET, derives most of its theoretical properties and details relevant applications. This representation is devoted to non-gaussian processes either stationary or not that exhibit non vanishing third order moments. It is also closely related to the deterministic correlation between the signal and its Wigner-Ville distribution which yields efficient implementation with almost the same numerical complexity as a Pseudo-Wigner-Ville representation.

1. INTRODUCTION

Great attention has been paid these last years to Wigner-Ville representations based on either higher order moments or cumulants symbolized by WHOS [1,2,3,4]. One of the main relevant difficulties is to reduce ,the dimensionality of any WHOS even in the lowest third order case to display in a three dimensional human world the non-stationarity of the underlying non-gaussian process. Although this issue is very crucial, there are only few works that tackle this problem. To some extent, the present paper proposes an alternative in the third order case but its scope is much larger since BET that we define and thorougly study hereafter has several interpretations. In this paper, only two of them have been chosen : 1) First of all, BET displays the third order properties of a stationary process in a time-frequency plane, which is sometimes very useful when these properties describe a frequency behaviour like for instance in the quadratic phase coupling. The structure of BET is not based upon an arbitrary building of what could be such a type of representation but on a necessary and analytical link in this stationary environment with both the bicorrelation and the bispectrum. 2) On the other hand, BET is nothing but the deterministic (*under* $l^2(Z)$ *sense*) intercorrelation between the signal itself and its Wigner-Ville distribution (WV) [5] which eases a lot the implementation. A third interpretation, related to non-stationary process, can be derived. BET may also be viewed as projection of the third order correlation or spectrum onto a particular manifold [6]. This interpretation, which is not discussed in this paper, justifies the use of BET to detect HOS changes like for instance gaussian to non-gaussian transitions.

The paper is organized as follows. In section 2, we give the definition of BET. Theoretical properties related to points 1 and 2 are derived and explained in section 3 for both continuous or discrete time processes and moreover in the context of the two interpretations. Section 4 is devoted to the implementation and the use of BET in the charaterization of quadratic phase coupling and non-gaussian phenomena.

2. THE BICORSPECTRAL TRANSFORM

BET is aimed to reveal the properties contained in the third order cumulants of a zero mean non-gaussian process in a time-frequency plane. Moreover, in our opinion, it appears very crucial to stress on these differences between the purpose of BET and any WHOS. This task will be accomplished in the next section. We only focus hereafter on the definition of BET.

In the whole paper, we assume that when x(t) is a continuous stochastic process, it is complex valued, zero mean, non-gaussian, with non vanishing and finite second and third order moments. When x(t) is considered as a deterministic signal (for instance when only a trial of a stochastic process is retained), that all the integrals where x(t) appears exist. The similar assumptions hold for a discrete time serie x[n], integrals are thus replaced by discrete summations.

With such assumptions, we define respectively the continuous and discrete versions of BET [6]

$$\beta_{xr}(t,f) \stackrel{\Delta}{=} \frac{1}{2T} \int_{\mathcal{R}^2} x_r(u-t)\, x_r\left(u+\frac{\tau}{2}\right) x_r^*\left(u-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau\, du \qquad (1)$$

$$\beta_{xr}[n,\nu] \stackrel{\Delta}{=} \frac{2}{2M-1} \sum_{(k,m)\in \mathcal{Z}^2} x_r[k-n]\, x_r[k+m]\, x_r^*[k-m]\, e^{-j4\pi\nu m} \qquad (1bis)$$

where xr is the windowed signal by the even window r of length 2T (or 2M-1 for the dicrete time version).

We will give two different interpretations of BET based on several properties established in the next section.

3. PROPERTIES OF BET

3.1. Properties of BET Related To The Third Order Statistics In The Stationary Case

In this section, statistical properties of BET are derived to show that it is a useful time-frequency representation to characterize third order phenomena even in the stationary case.

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S1. If x(t) is a continuous, zero mean, complex valued, stationary non-gaussian process, with non vanishing third order moments, then :

$$\lim_{T\to+\infty} E\{ \int_{\mathcal{R}} e^{+j2\pi f_2 t} \beta_{xr}(t,f_1+\frac{f_2}{2})\, dt \} = \overset{\circ}{\kappa}_x(f_1,f_2) \quad (2)$$

The proof of this property is given in the appendix A.

S1bis. if x[n] is a discrete time, zero mean, complex valued, stationary non-gaussian process, with non vanishing third order moments, and assume that :

$$\overset{\circ}{\kappa}_x(\nu_1,\nu_2)=0,\ (\nu_1,\nu_2)\in \{[-0.5,-0.25]U[0.25,0.5]\}^2$$

$$\lim_{N\to+\infty} E\{ \sum_{n\in Z} e^{+j2\pi\nu_2 n} \beta_{xr}[n,\nu_1+\frac{\nu_2}{2}]\} = \overset{\circ}{\kappa}_x(\nu_1,\nu_2),\ \text{for}$$

$$(\nu_1,\nu_2) \in \{[-0.25,\ 0.25]\}^2 \quad (2bis)$$

$\overset{\circ}{\kappa}_x(f_1,f_2)$ and $\overset{\circ}{\kappa}_x(\nu_1,\nu_2)$ are respectively the continous and discrete time bispectra.

The above derivation explains the use of the factor 2 in the definition of the discrete BET (1bis). The condition (2bis) required to allow the existence of property S1 in the discrete time case means, for instance, if the discrete signal has been obtained after sampling a continuous time signal, that the sampling rate used was twice the required rate, or that the initial process was replaced by its analytic signal.

S2. If x(t) is a continuous, zero mean, complex valued, stationary non gaussian process, with non vanishing third order moments, and define the Bicorspectrum as :

$$b_x(t,f) \overset{\Delta}{=} \int_{\mathcal{R}} \kappa_x(\tau,\frac{\tau}{2}-t)\, e^{-j2\pi f\tau}\, d\tau$$

with $\kappa_x(\tau_1,\tau_2)=E\{x(t+\tau_1),\ x(t+\tau_2),\ x(t)\}$

then : $\overset{\circ}{\kappa}_x(f_1,f_2) = \int_{\mathcal{R}} e^{+j2\pi f_2 t}\ b_x(t,f_1+\frac{f_2}{2})\, dt$

and : $\lim_{T\to+\infty} E\{\beta_{xr}(t,f)\}=b_x(t,f)=\int_{\mathcal{R}} \kappa_x(\tau,\frac{\tau}{2}-t)e^{-j2\pi f\tau}d\tau$

S3. if x(t) is a continuous, zero mean, complex valued, stationary non-gaussian process, with non vanishing third order moments, and define the indirect estimate of the Bispectrum called the Bicorrelogram as:

$$\overset{\circ}{\chi}_{xg}(f_1,f_2) \overset{\Delta}{=} \int_{\mathcal{R}^2} \chi_{xg}(\tau_1,\tau_2)e^{-j2\pi(f_1\tau_1+f_2\tau_2)}d\tau_1 d\tau_2$$

with : $\chi_{xg}(\tau_1,\tau_2)\overset{\Delta}{=}\frac{1}{2T}\int_{\mathcal{R}} x_g(v+\tau_1)x_g(v+\tau_2)\, x_g^*(v)dv$

then : $\overset{\circ}{\chi}_{xg}(f_1,f_2) \overset{a.s.}{=} \int_{\mathcal{R}} e^{+j2\pi f_2 t}\ \beta_{xg}(t,f_1+\frac{f_2}{2})\, dt$

and : $\beta_{xg}(t,f) \overset{a.s.}{=} \int_{\mathcal{R}} \chi_{xg}(\tau,\frac{\tau}{2}-t)\, e^{-j2\pi f\tau}\, d\tau$

where a.s. stands for almost surely.

With discrete time signal, we can derive such properties as S2 and S3 [6].

As a matter of fact, in the stationary case, the following crucial conclusions must be drawn :

- BET is a true time-frequency representation that reveals exactly all informations contained in the third order statistics of a zero mean complex valued non-gaussian stationary process (S2); its definition is not at all arbitrary since it relies on theorems that settles it "between" the Bicorrelation and the Bispectrum ;
- BET is (S2) an asymptotic unbiased estimate of the Bicorspectrum and may be viewed also, after a Fourier Transform, as an asymptotic unbiased estimate of the Bispectrum (S1), it is also asymptotically related to the Bicorrelation (S2) ;
- considering the same trial of a process, BET is almost surely related (S3) to the Bicorrelation estimate and the Bicorrelogram.

3.2. Properties of BET Related To The Bilinear Wigner-Ville Transform

In this section, we are interested in the properties of BET (1) (1bis) related to the classical bilinear Wigner-Ville transform [5] (by classical, we mean not depending upon higher order statistics). For this purpose, the signal x is assumed to be either deterministic or a unique trial of a stochastic process. We give hereafter the properties for a continuous signal, the extension to discrete time series is straightforward. All properties described below are mere consequences of definitions (1) and (1bis) and the property WV1 :

WV1. BET of a signal is proportional to the deterministic time intercorrelation in $L^2(\mathcal{R})$ or $l^2(\mathcal{Z})$ between the complex conjugate windowed signal and its Wigner-Ville transform [5], for any value of the frequency.

$$\beta_{xg}(t,f) \overset{\Delta}{=} \frac{1}{2T}\ (W_{xg}\ @\ x_g^-)(t,f)$$

where the convolution , denoted by @, is understood with respect to the time variable and x^- is the reverse time signal. Thus BET can be interpreted as a Pseudo-Wigner-Ville transform [5]. This interpretation makes easier the discretization and implementation proposed in section 4.

WV2. If a windowed signal takes non zero value only in the time interval $[t_0,\ t_1]$, then BET is non vanishing in the interval : $[-(t_1-t_0),\ (t_1-t_0)]$.

WV3. If x(t) is a real valued signal then :

$\beta_{xg}(t,f) = \beta_{xg}(t,-f)$ and $\beta_{xg}(t,f) = \beta_{xg}^*(t,f)$

WV4. Time shift invariance.

If: $\tilde{x}(t)\overset{\Delta}{=} x(t-t_0)$, then $\beta_{xg}(t,f) = \beta_{\tilde{x}g}(t,f)$.

This property is related to the non-stationary status of BET : if the time shift invariance vanishes, it means that a change has occurred in the signal.

WV5. Scale change invariance. If : $\tilde{x}(t)\ \overset{\Delta}{=} a^{2/3}\ x(\frac{t}{a})$, with $a \in \mathcal{R}^{+*}$, then $\beta_{xg}(t,f) = \beta_{\tilde{x}g}(t,f)$

WV6. Stability with respect to linear filtering.

$\tilde{x}(t) \triangleq (x @ h)(t)$, then $\beta_{\tilde{x}g}(t,f) = 2T\ (\beta_{xg} @ \beta_h)(t,f)$

where the convolution is understood with respect to the time variable.

Up to these properties, several differences between BET and WHOS can be drawn. First of all, althought both transforms have the same two kinds of arguments : time and frequency. But the number of arguments is not the same: WHOS requires one time instant and two frequencies [1,4]. Moreover, the structure of BET is not based on arbitrary building. Therefore, BET is not depending upon arbitrary choices of what could be a "good" higher order Wigner-Ville representation. This task is rather difficult as mentioned in [2].

4. SIMULATION

In this section, we are interested in an efficient implementation and we introduce applications which show the usefulness of this new representation.

The data we work with are assumed to be sampled and real. Moreover, the similarity between BET and PWV helps us to find an efficient implementation. Indeed, the implementation of BET is made by analogy with the second order PWV representation.

As we show in section 3 the analytic signal can be used to compute BET. Its use leads to write (1bis) as :

$$\beta_{xr}[n,\nu] \triangleq$$

$$= \frac{2}{2M-1} \Re\left\{ \sum_{m=-M+1}^{M} \sum_{k=k_1}^{k_2} z_x[k-n] z_x[k+m] z_x^*[k-m] e^{-j4m\pi\nu} \right\}$$

$k_1 = \min(-M+1,\ -M+1+n+m,\ -M+1+n-m)$

$k_2 = \max(M-1,\ M-1-n+m,\ M-1+n-m)$

z_x. is the analytic signal of x_r of the signal windowed by a rectangular window. $\Re\{.\}$ denotes the real part of a complex. The computational burden is equal to a PWV numerical complexity and BET can be straighforwardly computed with a 2.M-1 points Fast Fourier Transform FFT, by analogy with the PWV implementation [7]. Moreover, we note that this implementation is fit to a block data treatment, which suits with our BET computation.

The end of this paper is dedicated to the study of examples: the estimation of quadratic phase coupling and non-gaussian white noise. In some situations, when the signal is harmonic, non linear filters generate an harmonic component whose frequency and phase are the sum of the initial signal frequencies and phases. This phenomenon is called the quadratic phase coupling and can be detected with the third moment [2]. The classical quadratic phase coupling model is given by : $y(t) = \sum_{k=1}^{3} \exp j(2\pi\nu_k t + \Phi_k) + w(t)$

where w(t) is a zero-mean white Gaussian noise and $\nu_1 < \nu_2$, $\nu_3 = \nu_1 + \nu_2$; Φ_1 and Φ_2 are uniform independant random variables distributed on $[0,2\pi]$ and Φ_3 verifies :

$$\Phi_3 = \Phi_1 + \Phi_2 \quad (3)$$

Φ_3 is independent from Φ_1 and Φ_2 but Φ_3 follows the same law. The signal y is composed of three waves, but the relation (3) implies that the waves 1 and 2 have interacted in a quadratic manner [2]. For $0 < t < T$, BET expected values for the signal y(t) is :

$$E\{\beta yr(t,f)\} = \exp(j2\pi\nu_1 t) \frac{1-\cos(2\pi t(f-(\nu_1+\frac{\nu_2}{2})))}{T(2\pi(f-(\nu_1+\frac{\nu_2}{2})t))^2} + \exp(j2\pi\nu_2 t) \frac{1-\cos(2\pi t(f-(\nu_2+\frac{\nu_1}{2})))}{T(2\pi(f-(\nu_2+\frac{\nu_1}{2})t))^2}$$

The quadratic phase coupling signal BET is characterized by two sinusoids whose frequencies are ν_1 and ν_2 which appear at the frequencies $\nu_1+\frac{\nu_2}{2}$ and $\nu_2+\frac{\nu_1}{2}$. The figure 1 depicts an example of a quadratic phase coupling signal whose normalized frequencies ν_1 and ν_2 are respectively equal to 0,05 and 0,35.

When there is no quadratic phase coupling, we show that BET is nul [6]. The figure 2 depicts BET of sinusoidal signal without quadratic phase coupling. Thus, BET is a usefull representation to characterize the quadratic phase coupling phenomena. This result can also be reached by the bispectrum. However, when the the signal is not stationary, the problem of the quadratic phase coupling detection may not be solved by a bispectrum analysis. We are currently studying BET to non-stationary signals. The results of this study will be presented later.

Let us consider the problem of estimating a non-gaussian white noise process. It could be generated by passing a zero-mean gaussian white noise $\varepsilon(t)$ through a non linear filter : $w(t) = \varepsilon^2(t) - E[\varepsilon^2(t)]$. The second and the third order moments of a such signal are not vanished. Moreover, it can be shown that the espected value of BET is : $E[\beta_w(t,\nu)] = \kappa_3\delta(t)$, where κ_3 denotes the third order moment. The figure 3 illustrates a BET estimation of this non-gaussian signal. We notice that BET estimate does not fit with the theorical relation. It comes from the strong variance of this time-frequency estimator. The same phenomenon appears with the classical bispectrum estimator.

One of the aim of BET is to give a time-frequency representation of third order statistic of a stationary signal. The theoretical expressions are set up in this way. Nevertheless, we would like to use this mean in order to detect model breaking. For this purpose, we have to give up to the stationary propety. BET could be a candidat to analyse non-stationnary signals and could also be applied to detect gaussian to non-gaussien transitions or occurence of quadratic phase coupling.

5.CONCLUSION

In this paper, we have introduced, studied from a theoretical standpoint and applied to relevant problems a new time-frequency representation BET. It bears two interpretations. Fisrt of all it enables to display the third order properties of a non-gaussian stationary process in a time-frequency plane which is, to some extent, valuable for several non gaussian stationary phenomena that are related to harmonics. The last interpretation of BET, as a deterministic intercorrelation between the signal itself and its second order Wigner-Ville representation, yields a very efficient implementation that show precisely how the numerical complexity of BET is that of a Pseudo-Wigner-Ville representation.
Further works have already started that show properties of BET in non-stationary context. This later study provides that theorical standpoint of the using of BET for non-stationary signal.

REFERENCES

[1] J.Fonollosa, C.Nikias, "Wigner higher-order moment spectra: definition, properties, computation and application to transient signal analysis," IEEE trans. Signal Processing, Vol. SP-41, N°1, pp 245-266, 1993.
[2] P.O. Amblard, J.L. Lacoume, "A deductive construction of third-order time-frequency distributions," Signal Processing, Vol SP-36, N°3, pp 277-286,1994.
[3] B.Boashash, B. Ristich, "The wigner-Ville trispectrum : definition and application", Proc. IEEE Signal Proc. workshop on higher-order statistics, pp 260-264, South Lake Tahoe, Ca., USA, June 1993.
[4] G.Giannakis, A.Dandawate, "Polyspectral analysis of non-stationary signals : bases, consistency and HOS-WV, in higher order statistics", J.L.Lacoume ed., Elsevier, 1992.
[5] P.Flandrin, "Temps-fréquence," Hermès, 1993, ISBN 2-86601-387-5.
[6] P.Duvaut, T. Gouraud,"BET, The Bicorspectral Transform Definition, Properties, Implementation & Applications," Rapport interne ETIS, ETIS /GS/ 3N/ 9403.
[7] "Méthodes temps-frequence et segmentation de signaux," Traitement du signal, supplément Vol.9, N°1, 1992.
[8] F. Roddier, "Distribution et transformation de Fourier," MacGraw Hill, 1978.

APPENDIX A.

In this appendix, we derive the proof of the property S1. The other properties can be shown with similar proofs.
Proof. Starting from (1), considering the three variables : (τ, t', v) in such a way that : $\tau = \tau$, $t' = \frac{\tau}{2} - t$, $v = u - \frac{\tau}{2}$;
leads to :

$$E\{\int_{\mathcal{R}} e^{+j2\pi f_2 t}\,\beta_{xr}(t, f_1 + \frac{f_2}{2})\,dt\} =$$

$$= \int_{\mathcal{R}^2} \Psi_r(\tau,t')\,\kappa_x(\tau,t')\,e^{-j2\pi(f_1\tau+f_2t')}\,d\tau\,dt' \qquad \text{(A1)}$$

where $\kappa(\tau,t')$ is the Bicorrelation of $x(t)$ and where :

$$\Psi_r(\tau,t') = \frac{1}{2T}\int_{\mathcal{R}} r(v)r(v+\tau)r(v+t')\,dv \qquad \text{(A2)}$$

$\Psi_r(\tau,t')$ satisfies the 4 independent symetries :

$$\Psi_r(\tau,t') = \Psi_r(t',\tau) = \Psi_r(-\tau,t'-\tau) = \Psi_r(\tau-t',-t') = \Psi_r(-\tau,-t')$$

is equal to : $1-\frac{\tau}{2T}$, for $0 \le t' \le \tau$ (A4)

and has a double Fourier transform that converges to a Bidimensionnal Dirac distribution, according to the Fejer convergence [8] : $lim_{T\to+\infty}\,\mathcal{F}\{\Psi_r(\tau,t')\} = \delta(f_1)\delta(f_2)$ (A5)

The end of the proof comes from (A1) and (A5).

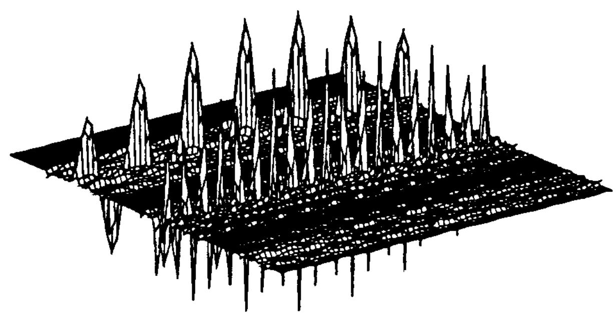

Figure 1 : BET of a quadratic phase coupling signal.

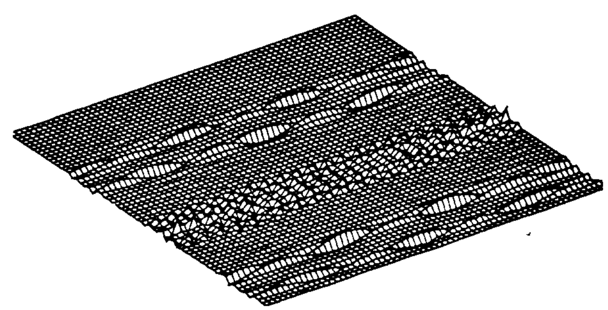

Figure 2 : BET of a signal without a quadratic phase coupling phenomenon.

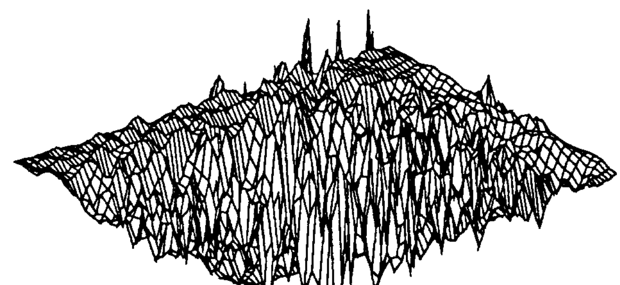

Figure 3 : BET of a white non-gaussian process.