HIGHER-ORDER SCALE SPECTRA AND HIGHER-ORDER TIME-SCALE DISTRIBUTIONS

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ABSTRACT

This paper develops a novel concept of higher-order moment and cumulant functions in the compress/stretch domain, and their corresponding higher-order spectra in the scale domain. Then higher-order Q time-scale distributions are introduced and their properties are investigated. The importance of the paper is to link the recently developed concept of scale signal representations with well established and important methods of higher-order spectral analysis.

1 INTRODUCTION

Scale domain signal analysis has been developed in response to a need to analyse scale-invariant signals such as Doppler-tolerant waveforms [12, Chap.12] or self-similar random processes [5]. Doppler tolerant waveforms are characterised by scale invariant instantaneous frequencies, while self-similar random processes are described as having scale-invariant probabilistic properties.

Recently Cohen [4] introduced the concept of scale transform, and joint time-scale representations. The scale transform of a deterministic transient (finite-energy) signal $x(t)$ for $t > 0$ represents the orthogonal form of the Mellin transform (MT) [2, p.254],

$$X(c) = \underset{t \to c}{\mathcal{M}}\{x(t)\} = \int_0^\infty x(t) \frac{e^{-j2\pi c \ln t}}{\sqrt{t}} dt \quad (1)$$

while one of the joint time-scale distributions is the Marinovich-Altes (or Q) distribution [9, 1]:

$$Q_x(t,c) = \int_0^\infty x(\tau^{1/2}\, t)\, x^*(\tau^{-1/2}\, t) \frac{e^{-j2\pi c \ln \tau}}{\tau} d\tau. \quad (2)$$

Variable $c \in \mathbb{R}$ is referred to as *scale*. According to [4], scale is as important signal attribute as it is frequency. In order to emphasise the importance of the concept of scale, many aspects of scale-domain analysis, such as the scale spectrum, the Wiener-Khinchin theorem, the uncertainty principle, have been considered so far [4].

Higher order statistical analysis has been developed in response to a need to examine phase relationships in signals, separate Gaussian and non-Gaussian processes or characterise non-linearities [10]. This paper develops a novel concept of higher-order moment and cumulant functions in the compress/stretch domain, and their corresponding scale higher-order spectra in the scale domain, following Cohen's concept of scale. The paper further introduces higher-order joint time-scale distributions and investigates their properties. The importance of the paper, thus, is to link the recently developed scale representations with well established and important methods of higher-order spectral analysis.

2 HIGHER-ORDER SCALE SPECTRA

2.1 Deterministic signals.

The k-th order moment function of $x(t)$ in the compress/stretch domain can be defined as:

$$m_x^{(k)}(\tau_1, \cdots, \tau_{k-1}) \triangleq \int_0^\infty x^*(t)\, x(\tau_1\, t) \cdots x(\tau_{k-1}\, t)\, dt, \quad (3)$$

and represents a numerical measure of the degree of similarity between a signal and a product of its stretched or compressed versions. Variables $\tau_i \in \mathbb{R}^+$ ($i = 1, \ldots, k-1$) are referred to as compress/stretch variables. If the k-th order scale spectrum of $x(t)$ is defined as:

$$M_x^{(k)}(c_1, c_2, \cdots, c_{k-1}) = \underset{\tau_1 \to c_1}{\mathcal{M}} \cdots \underset{\tau_{k-1} \to c_{k-1}}{\mathcal{M}} \{m_x^{(k)}(\tau_1, \cdots, \tau_{k-1})\},$$

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then we have:

$$M_x^{(k)}(c_1, c_2, \cdots, c_{k-1}) = X_{k-1}^*(c_1 + c_2 + \cdots + c_{k-1})\, X(c_1)\, X(c_2) \cdots X(c_{k-1})$$

where

$$X_n(c) = \underset{t \to c}{\mathcal{M}_n}\{x(t)\} = \int_0^\infty x(t) \frac{e^{-j2\pi c \ln t}}{t^{n/2}} dt \qquad (4)$$

represents a form of the MT. Note that $X(c)$ defined by (1), according to (4), corresponds to $X_{n=1}(c)$. In addition, the Q distribution in (2) can be written using $\mathcal{M}_{n=2}\{\cdot\}$ as:

$$Q_x(t, c) = \underset{\tau \to c}{\mathcal{M}_2}\{x(\tau^{1/2} t)\, x^*(\tau^{-1/2}\, t)\}.$$

Higher-order scale spectra are scale-invariant in the sense that $M^{(k)}_{x(b_0 t)}(\underline{c}) = M^{(k)}_{x(t)}(\underline{c})$, where $b_0 \in \mathbb{R}$, and $\underline{c} = [c_1, \cdots, c_{k-1}]$. For $k = 2$, one obtains the scale-spectrum $M_x^{(2)}(c_1)$ and the autocorrelation in the compress/stretch domain, $m_x^{(2)}(\tau_1)$ [4].

2.2 Random Signals

For a random signal $\mathbf{x}(t)$ we say that it is k-th order wide-sense self-similar ("scaling-stationary" [5]) if its k-th order moment function

$$m_{\mathbf{x}}^{(k)}(\tau_1, \cdots, \tau_{k-1}) = \mathrm{E}\{\mathbf{x}^*(t) \cdot \mathbf{x}(t\tau_1) \cdots \mathbf{x}(t\tau_{k-1})\} \qquad (5)$$

or cumulant function

$$c_{\mathbf{x}}^{(k)}(\tau_1, \cdots, \tau_{k-1}) = \mathrm{Cum}\{\mathbf{x}^*(t), \mathbf{x}(t\tau_1), \cdots, \mathbf{x}(t\tau_{k-1})\} \qquad (6)$$

defined in the compress-stretch domain does not depend on the time variable t. Operator $\mathrm{Cum}\{\cdot\}$ is defined in the usual manner [10, Sec.2.2.2]. It can be shown that the properties of cumulant functions of self-similar processes, defined in the compress/stretch domain, are analogous to those of the cumulant functions of stationary processes defined in the usual, time-shift, domain. For example:

Regions of symmetry. Both $m_{\mathbf{x}}^{(k)}(\tau_1, \cdots, \tau_{k-1})$ and $c_{\mathbf{x}}^{(k)}(\tau_1, \cdots, \tau_{k-1})$ are symmetric in their arguments, and hence the following regions of symmetry can be observed for real-valued $\mathbf{x}(t)$:

$$c_{\mathbf{x}}^{(2)}(\tau) = c_{\mathbf{x}}^{(2)}(\frac{1}{\tau})$$

$$c_{\mathbf{x}}^{(3)}(\tau_1, \tau_2) = c_{\mathbf{x}}^{(3)}(\tau_2, \tau_1) = c_{\mathbf{x}}^{(3)}(\frac{1}{\tau_2}, \frac{\tau_1}{\tau_2}) = c_{\mathbf{x}}^{(3)}(\frac{1}{\tau_1}, \frac{\tau_2}{\tau_1}) = c_{\mathbf{x}}^{(3)}(\frac{\tau_2}{\tau_1}, \frac{1}{\tau_1}) = c_{\mathbf{x}}^{(3)}(\frac{\tau_1}{\tau_2}, \frac{1}{\tau_2})$$

For real-valued $\mathbf{x}(t)$, if the scale spectrum is defined as $C_{\mathbf{x}}^{(2)}(c) = \mathcal{M}_{2\tau \to c}\{c_{\mathbf{x}}^{(2)}(\tau)\}$ then

$$C_{\mathbf{x}}^{(2)}(c) = C_{\mathbf{x}}^{(2)}(-c).$$

Similarly, if the scale bispectrum is defined as $C_{\mathbf{x}}^{(3)}(c_1, c_2) = \mathcal{M}_{2\tau_1 \to c_1} \mathcal{M}_{2\tau_2 \to c_2}\{c_{\mathbf{x}}^{(3)}(\tau_1, \tau_2)\}$, then it has the same twelve regions of symmetry as the conventional bispectrum [11, p.390].

Scalar measures. The scalar measures of a real-valued random signal $\mathbf{x}(t)$ are obtained from $c_{\mathbf{x}}^{(k)}(\tau_1, \cdots, \tau_{k-1})$ by taking $\tau_1 = \cdots = \tau_{k-1} = 1$: variance ($k = 2$), skewness ($k = 3$), kurtosis ($k = 4$), etc.

Example. Estimates of the scale-spectrum and the normalised scale-bispectrum of a natural signal (bat sonar echolocation pulse) are shown in Fig.1. This signal is *non-stationary*, but it appears that it is third order self-similar (i.e. its scale-spectrum and the scale-bispectrum are time-invariant). The peak in the normalised scale-bispectrum indicates the likely quadratic phase coupling between two signal components (represented by two strong and sharp peaks in the scale-spectrum). Details of scale domain analysis of this bat signal can be found in [13].

Other examples of self-similar processes are the Wiener-Lévy process [11], [5] and the fractional Brownian motion [8].

3 HIGHER-ORDER Q TIME-SCALE DISTRIBUTION

3.1 Definition

The higher-order Q distribution (HO-QD) will be introduced as a scale-invariant counterpart of the higher-order Wigner-Ville distribution (HO-WVD), recently defined and studied by Gerr [7] and Fonollosa and Nikias [6]. We define higher-order Q distributions as:

$$Q_x^{(k)}(t, c_1, \cdots, c_{k-1}) = \underset{\tau_1 \to c_1}{\mathcal{M}_2} \cdots \underset{\tau_{k-1} \to c_{k-1}}{\mathcal{M}_2} \{x(a_0 t)^*\, x(a_1 t) \cdots x(a_{k-1} t)\}$$

where a_0, a_1, $\cdots$, a_{k-1} are functions of compress/stretch variables τ_1, τ_2, $\cdots$, τ_{k-1}, and must satisfy the following two constraints:

<u>Constraint 1:</u> $\qquad a_i / a_0 = \tau_i \qquad (i = 1, \cdots, k-1) \qquad (7)$

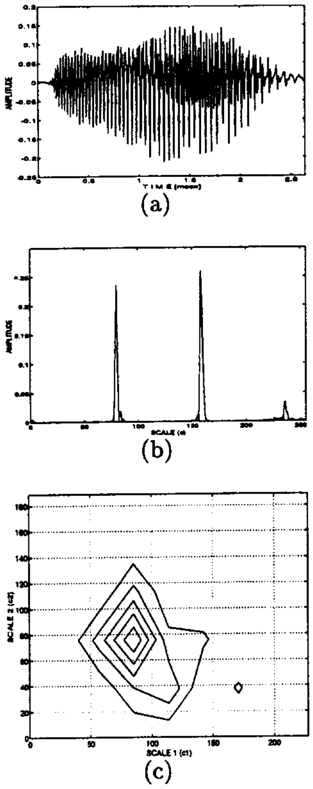

Figure 1: (a) The bat echo-locating signal (temporal representation); (b) its scale spectrum and (c) its normalised scale bispectrum

Constraint 2: $a_0 \cdots a_1 \cdots a_{k-1} = 1$ (8)

Constraint 2 is also referred to as compress/stretch "centering". This choice of constraints will ensure that higher-order Q distributions preserve and generalise many desirable properties of the Q distribution. From (7) and (8), it follows that:

$$a_0 = \left(\prod_{i=1}^{k-1} \tau_i \right)^{-1/k},$$

and the expression for higher-order Q distribution becomes:

$$Q_x^{(k)}(t, c_1, \ldots, c_{k-1}) = \int_{\underline{\tau}=\underline{0}}^{\infty} x^*\left(\frac{t}{(\prod_{i=1}^{k-1} \tau_i)^{1/k}}\right) \prod_{i=1}^{k-1} x\left(\frac{t\tau_i^{\frac{k-1}{k}}}{(\prod_{l=1;l\neq i}^{k-1} \tau_l)^{1/k}}\right) \frac{e^{-j2\pi c_i \ln \tau_i}}{\prod_{i=1}^{k-1} \tau_i} d\tau_i \quad (9)$$

where $\underline{\tau} = [\tau_1, \ldots, \tau_{k-1}]$. For $k = 2$, eq.(9) amounts to the definition of the conventional Q distribution given by (2). For $k = 3$ eq.(9) represents the third order Q distribution, given by:

$$Q_x^{(3)}(t, c_1, c_2) = \int_0^{\infty} \int_0^{\infty} x^*(t\tau_1^{-1/3}\tau_2^{-1/3}) \, x(t\,\tau_1^{2/3}\,\tau_2^{-1/3}) \, x(t\,\tau_1^{-1/3}\,\tau_2^{2/3}) \frac{e^{-j2\pi(c_1 \ln \tau_1 + c_2 \ln \tau_2)}}{\tau_1\tau_2} d\tau_1 d\tau_2.$$

3.2 Some properties of HO-QD

Relationship to the higher-order Wigner-Ville distribution. The HO-WVD was defined as [6]:

$$W_x^{(k)}(t, \underline{f}) = \int_{\underline{\tau}=\underline{-\infty}}^{\underline{\infty}} x^*(t - \frac{1}{k}\sum_{i=1}^{k-1} \tau_i) \prod_{i=1}^{k-1} x(t + \frac{k-1}{k}\tau_i - \frac{1}{k}\sum_{l=1;l\neq i}^{k-1} \tau_l) \, e^{-j2\pi f_i \tau_i} d\tau_i$$

with $\underline{f} = [f_1, \cdots, f_{k-1}]$ and with limits of integral from $-\infty$ to ∞. It is related to HO-QD as follows:

$$W_{\{x(e^t)\}}^{(k)}(t, \underline{c}) = Q_{\{x(t)\}}^{(k)}(e^t, \underline{c}).$$

Compress / stretch covariance.

$$Q_{\{x(b_0 t)\}}^{(k)}(t, \underline{c}) = Q_{\{x(t)\}}^{(k)}(b_0 t, \underline{c}) \qquad (b_0 \in \mathbb{R}^+)$$

Scale shift covariance. If the HO-QD is defined with every second signal term complex-conjugated, i.e.

$$Q_x^{(k)}(t, c_1, \ldots, c_{k-1}) = \int_{\underline{\tau}=\underline{0}}^{\infty} x^*\left(\frac{t}{(\prod_{i=1}^{k-1} \tau_i)^{1/k}}\right) \prod_{i=1}^{k-1} \mathcal{P}_i\{x(\frac{t\tau_i^{\frac{k-1}{k}}}{(\prod_{l=1;l\neq i}^{k-1} \tau_l)^{1/k}})\} \frac{e^{-j2\pi c_i \ln \tau_i}}{\prod_{i=1}^{k-1} \tau_i} d\tau_i \quad (10)$$

where $\mathcal{P}_i\{\}$ is a complex-conjugate operator if i is even and identity operator otherwise, then for even valued k we have for $c_0 \in \mathbb{R}$:

$$Q_{\{x(t)\exp(j2\pi c_o \ln t)\}}^{(k)}(t, \underline{c}) = Q_{\{x(t)\}}^{(k)}(t, c_1 - c_0, c_2 + c_0, \ldots, c_{k-1} - c_0).$$

Time marginal. For HO-QD defined in (9)

$$\int_{\underline{c}} Q_x^{(k)}(t,\underline{c})d\underline{c} = x^*(t)\,x^{k-1}(t).$$

For definition (10) and even-valued k:

$$\int_{\underline{c}} Q_x^{(k)}(t,\underline{c})d\underline{c} = |x(t)|^k.$$

Multiplication property. The HO-QD of a product of two signals $u(t)\,v(t)$ yields:

$$\begin{aligned} Q_{u\cdot v}^{(k)}(t,\underline{c}) &= \int_{\underline{\eta}} Q_u^{(k)}(t,\underline{\eta})\,Q_v^{(k)}(t,\underline{c}-\underline{\eta})d\underline{\eta} \\ &= Q_u^{(k)}(t,\underline{c}) \underset{(\underline{c})}{*} Q_v^{(k)}(t,\underline{c}) \end{aligned}$$

where $\underset{(\underline{c})}{*}$ denotes convolution in the multi-scale space.

Mean conditional scale and the instantaneous scale. The mean scale of the HO-QD over the multi-scale space, at each time, is proportional to the *instantaneous scale* of signal $x(t)$ [3] $c_i(t) = \frac{1}{2\pi}t\frac{d}{dt}\{\arg[x(t)]\}$, i.e. for $m = 1,\dots,k-1$ we have:

$$\frac{\int_{\underline{c}} c_m Q_x^{(k)}(t,\underline{c})d\underline{c}}{\int_{\underline{c}} Q_x^{(k)}(t,\underline{c})d\underline{c}} = \frac{2}{k}c_i(t) \tag{11}$$

Alternative form. The HO-QD can be expressed using the MT of signal $x(t)$ as follows:

$$Q_x^{(k)}(t,\underline{c}) =$$
$$\frac{1}{t^{k/2}}\int_\gamma X^*\left(\sum_{i=1}^{k-1} c_i - \frac{\gamma}{k}\right)\prod_{l=1}^{k-1} X(c_l + \frac{\gamma}{k})\,e^{j2\pi\gamma\ln t}d\gamma$$

Then it can be shown that properties such as the scale marginals or the convolution property are not satisfied. For example:

$$\int_t t^{\frac{k-2}{2}} Q_x^{(k)}(t,\underline{c})dt = X^*(\sum_{i=1}^{k-1} c_i)\prod_{i=1}^{k-1} X(c_i).$$

In order to satisfy these properties, one would have to modify the proposed definition of the HO-QD and to sacrifice the multiplication property and the mean conditional scale property.

A general class of higher-order time-scale distributions. Higher-order time-scale distributions can be defined as:

$$P_x^{(k)}(e^t,\underline{c}) = Q_x^{(k)}(e^t,\underline{c}) \underset{(t)}{*}\underset{(\underline{c})}{*} \phi(e^t,\underline{c}),$$

where $\phi(t,\underline{c})$ is the kernel in the time-multiscale domain, and $\underset{(t)}{*}$ denotes the convolution in time.

4 SUMMARY

The paper has introduced higher-order scale spectra and discussed some of their basic properties. The concept of higher-order Q and general time-scale distributions has been developed based on the "centering" condition of compress/stretch variables. The resulting higher-order Q distribution preserves or generalises many of the properties of the Q distribution.

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