A CLASS OF SECOND ORDER SELF-SIMILAR PROCESSES FOR $1/f$ PHENOMENA[1]

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ABSTRACT

In this paper, we introduce a class of stochastic processes whose correlation function obeys a structure of the form, $E[X(t)X(\lambda t)] = t^{2H}\lambda^{H}R(\lambda)$, $\lambda, t > 0$. We refer to these processes as second order self-similar processes. This class of processes include fractional Brownian motion as a special case. We define a concept of autocorrelation and develop a spectral analysis framework via generalized Mellin transform for the proposed class. Additionally, we establish a relationship between the proposed self-similar processes and generalized linear scale invariant system theory. We give specific models and demonstrate their ability to model $1/f$ phenomena.

I. Introduction

$1/f$ physical phenomena are often identified by their measured Fourier spectra obeying a power law of the form, $S(f) \propto 1/|f|^{\beta}$, $\beta > 0$. They are typically characterized by an inherent statistical self-similarity and relatively strong correlations between far apart observations. Therefore, conventional models, such as ARMA models provide poor representations for such empirical findings. In general, the construction of long term correlated processes are either based on the physical origins of the $1/f$ noise, such as the ones proposed by Ziel [1] and Keshner [2] or based on the concept of statistical self similarity, a form of invariance with respect to changes of time scale. Fractional Brownian motion (fBm) proposed by Mandelbrot and Van Ness [3] is an example of the latter one. It can be viewed as a moving average of the white noise in which past increments are weighted by the kernel $(t-s)^{H-1/2}$, $0 < H < 1$. While fBm has become a popular framework in variety of engineering problems involving $1/f$ processes, it has a number of limitations. In particular, due to its nonstationary structure, as yet there is no satisfactory spectral analysis framework. Additionally, fBm is a Gaussian process and can only be used as a covariance model. There is a need, however, for non-Gaussian, generative models [4].

We introduce a class of self-similar processes for $1/f$ phenomena that includes fBm as a special case, yet avoids the restrictions described above. We define a concept of autocorrelation for the proposed self-similar processes and developed a spectral analysis framework via generalized Mellin transform. Additionally, we establish a relation between the proposed self-similar process and the generalized linear scale invariant system theory. We give examples of specific models and demonstrate their ability to represent long-term correlations.

II. Proposed Class of Models

The long run properties of the phenomena exhibiting $1/f$ spectra can be parsimoniously modeled by self-similar processes [3]. A stochastic process $\{X(t), -\infty < t < \infty\}$ is called statistically self-similar with parameter H, if it satisfies the following scaling condition

$$X(t) \equiv a^{-H}X(at), \quad a > 0 \quad -\infty < t < \infty, \qquad (2.1)$$

where $\equiv$ denotes the equality in terms of finite joint distributions. We interpret the statistical self similarity in

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terms of second order statistics in which the equality in (2.1) is alternately expressed as

$$E[X(t)] = a^{-H} E[X(at)]$$

$$E[X(t_1)X(t_2)] = a^{-2H} E[X(at_1)X(at_2)],$$

$$a>0 \quad -\infty < t, t_1, t_2 < \infty. \tag{2.2}$$

For all practical purposes, we shall restrict our attention to second order self-similar processes with finite variance. For $H = 0$, second order self-similar processes with finite variance possesses structural properties that are similar to classical wide sense stationary processes. Moreover, they lead to a rich class of self-similar processes with arbitrary parameter H. Therefore, we propose the following definition.

Definition 1: A random process $\{X(t), t>0\}$ shall be said to be scale-stationary if it satisfies the following conditions.

i) $E[X(t)] = \text{constant}$, for all $t \in (0,\infty)$.

ii) $E\left[|X(t)|^2\right] < \infty$, for all $t \in (0,\infty)$.

iii) $E\left[X(t_1)\overline{X(t_2)}\right] = E\left[X(\alpha t_1)\overline{X(\alpha t_2)}\right]$ for all $t_1, t_2, \alpha > 0$ □

Scale-stationary processes allows us to construct finite variance, self-similar processes with arbitrary parameter H. Such a construction is given by the following result, which is a straightforward corollary to Definition 1.

Theorem 1: Define $X(t) = t^H \tilde{X}(t)$, $t>0$. Then, $\{X(t),\ t>0\}$ is self-similar with finite variance and parameter H if and only if $\{\tilde{X}(t),\ t>0\}$ is scale-stationary. □

We shall call $\{\tilde{X}(t),\ t>0\}$, the generating scale-stationary process of $\{X(t),\ t>0\}$. It is clear that the correlation function of a self-similar process is related to its generating process with the following expression.

$$E[X(\lambda t)X(t)] = (\lambda t)^H t^H E[\tilde{X}(\lambda t)\tilde{X}(t)] = t^{2H}\lambda^H R_{\tilde{X}}(\lambda),$$
$$t,\lambda > 0 \tag{2.3}$$

where $R(\cdot)$ is a positive definite function on the multiplicative group. In particular, for $t = 1$,

$$E[X(\lambda)X(1)] = \lambda^H R_{\tilde{X}}(\lambda) \equiv \Gamma_X^H(\lambda), \quad \lambda > 0. \tag{2.4}$$

We shall call $\Gamma_X^H(\cdot)$ the *basic autocorrelation function* of the self-similar process $\{X(t),\ t>0\}$ with parameter H. Note that $E[X(\lambda t)X(t)] = t^{2H}\Gamma_X^H(\lambda)$, $t,\lambda>0$. The basic autocorrelation function $\Gamma_X^H(\cdot)$ represents the underlying stationary structure of the finite variance self-similar processes defined on the positive axis. The following theorem shows that the basic autocorrelation function is sufficient to characterize a finite variance, self-similar process defined on the positive axis.

Theorem 2: A function $\Gamma_X^H(\cdot)$ defined on the positive real axis is the basic autocorrelation function of a self similar process with the parameter H if and only if there exists a nonnegative symmetric measure F on $(-\infty,\infty)$ so that

$$\Gamma_X^H(\lambda) = \int_{-\infty}^{\infty} \lambda^{j\omega+H} dF(\omega), \quad \lambda > 0. \tag{2.5}$$

Proof: See [5]. □

As a consequence, the process has the following spectral representation

$$X(t) = \int_{-\infty}^{\infty} t^{j\omega+H} dB(\omega), \quad t > 0. \tag{2.6}$$

where the integral is defined in the mean square sense and $B(\omega)$ is the Brownian motion. Moreover, if F is absolutely continuous, we have

$$S(\omega) = \frac{dF}{d\omega}(\omega) = \int_0^{\infty} \lambda^{j\omega-H-1}\Gamma_X^H(\lambda) d\lambda. \tag{2.7}$$

In the familiar terminology of engineering, the above theorem simply states that generalized Mellin transform with parameter H, whitens the finite variance self-similar process with parameter H. As is well known, the output of the linear time invariant systems driven by a wide sense stationary processes, is also wide sense stationary. The following theorem proves the counterpart of this fact for linear scale invariant systems (LSI) [6] and finite variance self-similar processes.

Theorem 3: Let $h(\cdot)$ be the impulse response of a LSI system with parameter H_1 and $\{x(t), t>0\}$ be the self similar process with parameter H_2. If $h(t)t^{-(H_1+H_2)}$ is in the space $L^2((0,\infty),dt/t)$, then the output

$$y(t)=\int_0^\infty h\left(\frac{t}{\lambda}\right)x(\lambda)\frac{d\lambda}{\lambda^{1-H_1}}, \text{ for all } t>0 \tag{2.8}$$

is a finite variance, self-similar process with parameter H_1+H_2.

Proof 3 : See [5]. □

III. Examples

Example 1: It is well-known that the ARMA processes

$$\alpha_N t^N \frac{d^N}{dt^N}y(t)+\cdots+\alpha_1 t\frac{d}{dt}y(t)+\alpha_0=\beta_M t^{M+H}\frac{d^M}{dt^M}x(t)+\cdots+\beta_1 t^{1+H}\frac{d}{dt}x(t)+t^H x(t) \tag{3.1}$$

are obtained by driving the linear time invariant system whose dynamics are represented by constant coefficient ordinary differential equations by the "white noise" processes. Similarly, there is a class of ordinary differential equations, known as Euler-Cauchy system, that is suitable to represent the dynamics of the LSI systems with parameter 0 [6]. We could modify Euler-Cauchy system to obtain a class of time varying differential equations to represent the dynamics of the LSI system with arbitrary parameter H. We can show rigorously that the LSI system represented by the Equation (3.1) yields a self-similar process with parameter H, when it is driven by the "white noise". We call this process self-similar autoregressive moving average process with parameter H (SS-ARMA(H)). For H = 0 and M, N = 1, the basic correlation of the process is given by

$$E[X(t)X(t\lambda)]=\begin{cases}\lambda^{-\alpha} & \lambda\geq 1\\ \lambda^{\alpha} & 0<\lambda<1\end{cases}. \tag{3.2}$$

Example 2 : Let φ be a uniformly distributed random variable on $(-\pi,\pi)$. Consider the following process.

$$x(t)=at^H\cos(\omega_0\ln t+\varphi) \tag{3.3}$$

We can show by direct calculation that $\{x(t),t>0\}$ is a self similar process and its basic correlation function is given by $\Gamma_x^H(\lambda)=(\sigma^2/2)\lambda^H\cos(\omega_0\ln\lambda)$.

IV. Long-term Correlations and Self-Similarity

It is well-known that not all self-similar processes are long term correlated. For our proposed self-similar process to be long term dependent, we require the following condition: The sum of the correlations, $E[X(t+\tau)X(t)]$, has to be infinite for each fixed $t>0$, i.e.,

$$t^{2H}\int_{-t}^{\infty}\left(1+\frac{\tau}{t}\right)^H R_{\tilde{X}}\left(1+\frac{\tau}{t}\right)d\tau\to\infty, \tag{4.1a}$$

or alternatively,

$$\int_0^\infty \Gamma_X^H(\lambda)d\lambda\to\infty. \tag{4.1b}$$

This condition ensures relatively slow decay of the correlations as the lag $\tau\to\infty$. Our simulation experiments justifies the correlation criterion chosen.

V. Conclusion

In this paper, we introduced a class of self-similar processes that is particularly well-suited for engineering applications involving $1/f$ processes. The Figure shows sample paths of various finite variance, self-similar processes defined on the positive axis. The proposed analysis framework suggests new signal processing methods for problems involving $1/f$ phenomena.

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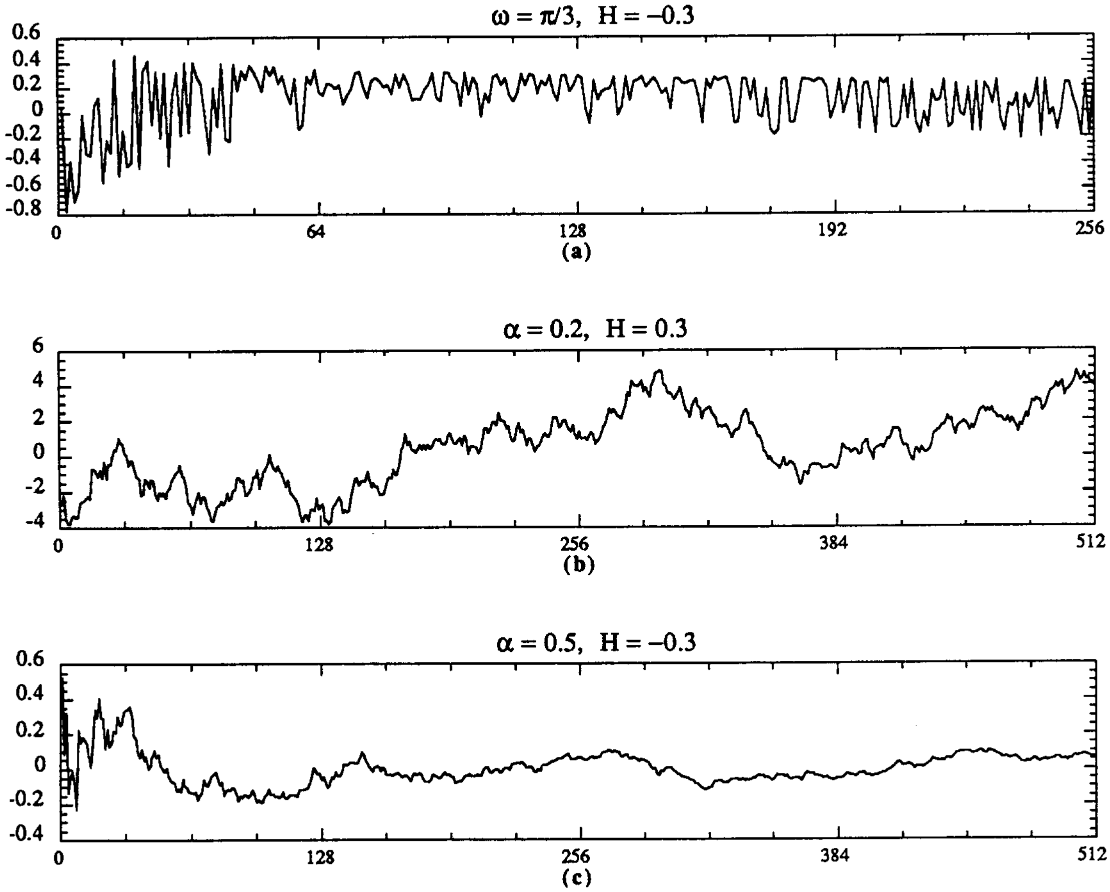

The Figure

Sample paths of various finite variance, self-similar processes defined on the positive real axis: (a) $x(t) = \sigma t^H \cos(\pi/3 \ln t + \varphi)$, φ uniform on $(-\pi, \pi)$, H = -0.3, (b&c) First order SS-AR(H) process, (b) $\alpha = 0.2$, H = 0.3, (c) $\alpha = 0.5$, H = -0.3.