DISCRETE SCALE TRANSFORM FOR SIGNAL ANALYSIS

Eugene J. Zalubas and William J. Williams

Department of Electrical Engineering and Computer Science
The University of Michigan, Ann Arbor, MI 48105-2122
email: loz@eecs.umich.edu

ABSTRACT

The scale transform introduced by Cohen [1] is a special case of the Mellin transform. The scale transform has mathematical properties desirable for comparison of signals for which scale variation occurs. In addition to the scale invariance property of the Mellin transform many properties specific to the scale transform have been presented[1]. A procedure is presented in this paper for complete implementation of the scale transformation for discrete signals. This complements discrete Mellin transforms and delineates steps whose implementation are specific to the scale transform.

1. INTRODUCTION

Signal scaling arises in many situations. For example, a waveform sampled at different rates will generate discrete signals which differ, but are related by a scale factor. Scale changes are induced by a number of real world phenomena such as the Doppler effect.

The Mellin transform is defined as [3]

$$F(s) = \int_0^\infty f(t)t^{s-1}dt \tag{1}$$

The scale transform is the specific case where $s = 1/2 - jc$. An equivalent definition

$$D(c) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\frac{e^{-jc\ln t}}{\sqrt{t}}dt \tag{2}$$

presented by Cohen [2] yields the scale transformation, $D(c)$, of the time domain signal $f(t)$. The scale transform may also be viewed as the Fourier transform of the function $f_k(t) = f(e^t)e^{t/2}$ [1].

$$D(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f_k(t)e^{-jct}dt \tag{3}$$

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Exponential distortion transforms a scale factor in the time variable to a translation in scale. That is, the energy normalized scaled signal $\sqrt{a}f(at)$ yields $\sqrt{a}f(ae^t)e^{t/2} = f(e^{t+\ln a})e^{(t+\ln a)/2} = f_k(t+\ln a)$. The Fourier transform in equation (3) converts the translation term into a phase factor. Two signals normalized to equal energy which differ only in scale have coefficients of scale identical within a phase factor. Thus, the scale invariance property of the scale transform permits direct comparison of signals in which scale is not consistent.

Zwicke and Kiss [4] presented two discrete implementations of the Mellin transform. Since properties of the Mellin transform vary as a function of s in equation (1), detailed discrete implementations for all Mellin transforms cannot be presented in a unified manner. An implementation for a specific case of the scale transform has not been presented. In this paper, a complete implementation of a discrete scale transform is presented. A method for resampling uniformly spaced samples to exponential spacing is included.

2. DISCRETE IMPLEMENTATION OF SCALE TRANSFORM

Signals are commonly sampled at uniformly spaced intervals. Implementation of the scale transform in equation (3) requires exponential sampling of $f(t)$ to yield a uniformly spaced $f_k(t)$. Since discreteness of the data is assumed, interpolation is required to find the signal value at the exponentially spaced locations. For exact results under DFT sampling assumptions, sinc interpolation is the correct method. All other interpolation methods will modify the existing frequency components or introduce extraneous ones.

A signal uniformly sampled at or above the Nyquist rate may be exponentially sampled using the following method.

1. Select the exponentially spaced sample locations. Assuming that the original signal is sampled at

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the Nyquist rate, the maximum exponential intersample distance must not exceed the uniform sample spacing. For an original signal consisting of N data points, the exponentially sampled version is adequately sampled using $N \log N$ points.[5]

Exponentially spaced locations may be selected by setting $t = N$ and $t = N - 1$ as sample locations. The geometric factor between these locations is $k = N/(N-1)$. To obtain the previous location, divide $N-1$ by the factor, k. Divide again by k to obtain another sample location. Continue this procedure until a total of $N \log N$ exponentially spaced sample locations are found.

2. Using the FFT, calculate the Discrete Fourier Transform (DFT) of the samples, f(n), of the original signal:

$$F(k) = \sum_{n=0}^{N-1} f(n) e^{\frac{-j2\pi kn}{N}}. \tag{4}$$

   The resulting $F(k)$ are equally spaced samples of the frequency domain representation of the signal [6].

3. Interpolate the signal using the coefficients produced by the FFT. These coefficients define a unique signal in continuous-time. The continuous-time signal, $\tilde{f}(t)$, is reconstructed by

$$\tilde{f}(t) = \sum_{k=0}^{N-1} F(k) e^{\frac{j2\pi kt}{N}}. \tag{5}$$

   The reconstructed signal, $\tilde{f}(t)$, has the same value as $f(n)$ at locations where $t = n$ for $0 < n < N$. Values of $\tilde{f}(t)$ for locations between the coincident points are perfectly interpolated with sinc functions. This reconstruction implicitly assumes that $f(n) = f(n \bmod N)$.

   Exponentially spaced samples of $\tilde{f}(t)$, are desired. The inverse FFT yields equally spaced samples and, therefore, cannot be used. The summation in equation (5) must be performed for each of the $N \log N$ exponentially spaced sample locations. Thus, this is a computationally intensive procedure. However, fast algorithms for the entire procedure are of great interest and should be pursued.

4. Evaluate the signal multiplied by $e^{t/2}$ at the exponentially spaced sample points of step 1. The result is treated as the uniformly spaced signal $f_k(n)$ which represents a discrete version of the $f_k(t)$ in equation (3). This relabeling of the signal provides the exponential distortion of time that yields the scale invariance property.

5. Obtain the scale domain representation $D(c)$ by performing a DFT on $f_k(n)$. Since the samples, $f_k(n)$, are considered to be uniformly spaced, an FFT may used in place of the DFT for faster processing.

6. Account for the contribution of the signal at time zero. $f(0)$ corresponds to $f_k(-\infty)$. Since $f_k(t) = f(e^t)e^{t/2}$, it is evident in the continuous case that the time domain signal makes no contribution at time zero. In the discrete case, however, each sample represents a finite time interval. $f(0)$ of the original data sequence represents the signal between time zero and the first exponential sample point. Since this interval has a non-zero weighting in the computation of $f_k(t)$, the contribution of the value at time zero must be included.

   Using the calculation method of [4], $D_0(c)$ the contribution of $f(0)$ to the $D(c)$ value may be found. Unfortunately, a simple result does not appear to be evident. For each scale, the value required to provide scale coefficients of equal magnitude for square pulses of varying length must be computed. The result is dependent on the number of exponential samples used.

   The $D_0(c)$ value is added to the $D(c)$ values already computed to achieve the final result. This final result is the discrete scale transform of the original signal samples, $f(n)$.

3. EXAMPLES

We present two examples which illustrate the operation of the Fourier based discrete scale transform. In each case, a root signal is compared to a scaled version of the root signal. The first signal examined is a single cycle sinusoid. The second signal is a square pulse.

3.1. Single Cycle Sinusoid

Consider the single cycle sinusoid and scaled version depicted in figure 1.

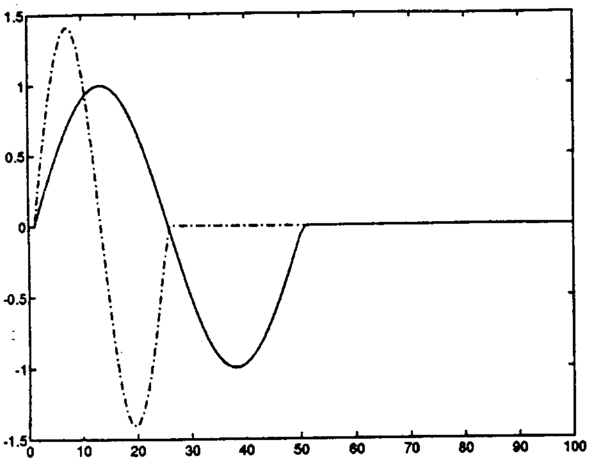

Figure 1: Single cycle sinusoid (solid) and scaled version (dashed)

The exponentially sampled versions of the signals are shown in figure 2. It requires 462 exponentially sampled points to represent the 100 point original signal.

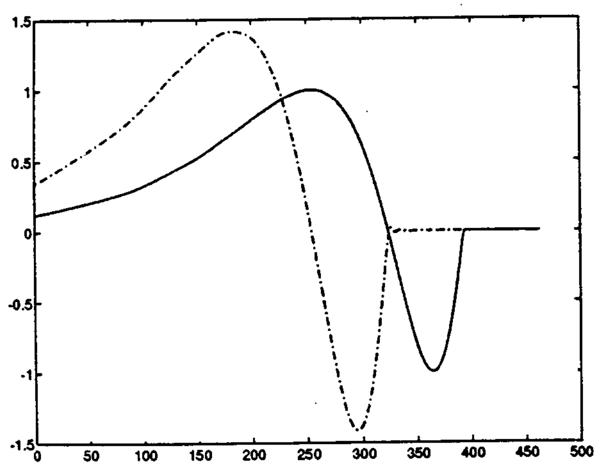

Figure 2: Exponentially sampled single cycle sinusoid (solid) and scaled version (dashed). Horizontal axis is linear in sample number to show exponential nature of sampling

The scale transform magnitudes calculated by the discrete method described above are depicted in figure 3. Slight variation occurs due to discrete sampling and implicit assumptions associated with the FFT.

3.2. Square pulse

In this example, two square pulses of equal energy, beginning at time $t = 0$ are compared. Figure 4 shows the signals and the exponentially sampled versions. The

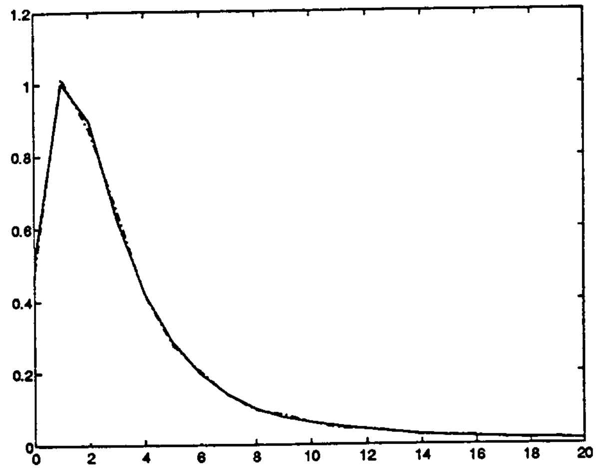

Figure 3: Magnitude of the scale transform for the single cycle sinusoid (solid) and its scaled version (dashed)

analytical result for the scale transform of a unit magnitude square pulse beginning at time zero and ending at time t_0 is

$$| D(c) | = \frac{t_0^{\frac{1}{2}}}{\sqrt{2\pi(\frac{1}{4} + c^2)}} \tag{6}$$

Consider the square pulses depicted in figure 4. The exponentially sampled versions shown in figure 5.

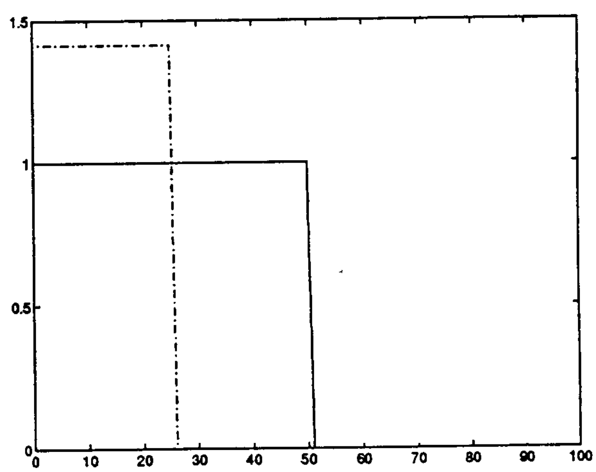

Figure 4: Square pulse (solid) and scaled square pulse (dashed)

In the exponentially sampled signals there is significant "ringing" due to the implicit assumption that the signal is bandlimited. Because the exponential sampling method utilizes the DFT, there is the implicit assumption that the signal is bandlimited and periodic in the length of the window. Thus, the exponentially sampled versions of the signal have evidence of periodic components related to the length of the window.

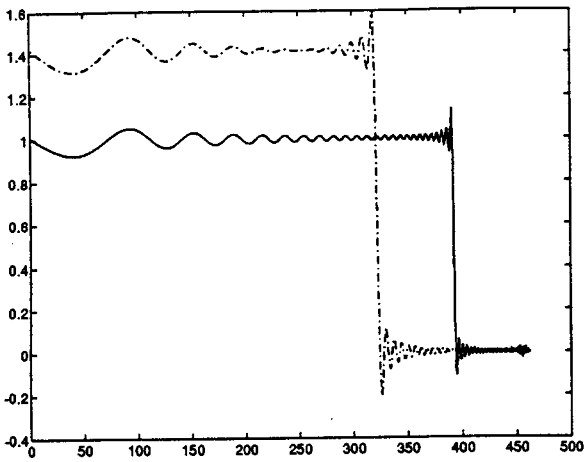

Figure 5: Exponentially sampled square pulse (solid) and scaled square pulse (dashed). Horizontal axis is linear in sample number to show exponential nature of sampling

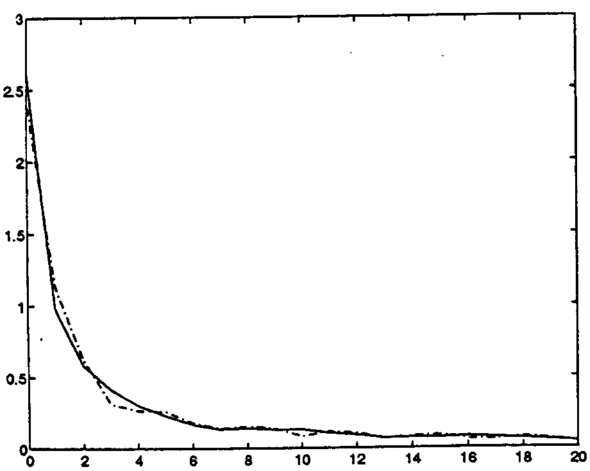

Figure 6: Magnitude of the scale transform for the square pulse (solid) and scaled square pulse (dashed)

Magnitudes of the scale transform of these two signals are very similar. The phase, however, can be used to differentiate between signals of the same structure. Figure 7 shows the real part of the scale transforms for the square pulses.

4. CONCLUSIONS

The scale transform represents the scaling of a signal as a phase shift. A discrete implementation for the scale transform has been presented. This straightforward implementation brings scale invariance property of the Mellin transform and specific desirable properties of the scale transform [1] to discrete data.

The rapidly evolving concepts of scale open the door to many interesting and useful theoretical developments and applications. In order to take advantage of this, a viable discrete scale transform is required.

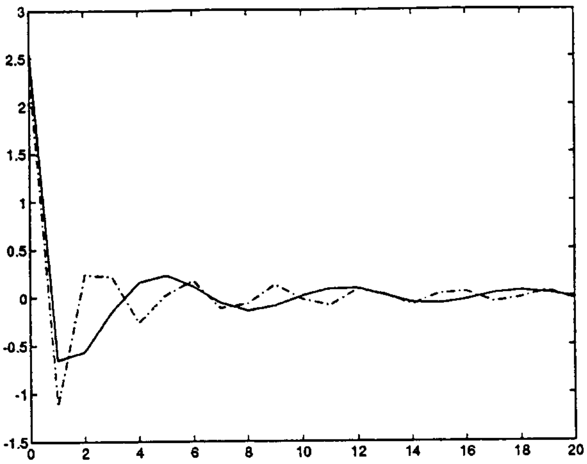

Figure 7: Real part of the scale transform for square pulse (solid) and scaled square pulse (dashed)

5. REFERENCES

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