VARIABLE FREQUENCY EVOLUTIONARY SPECTRUM

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ABSTRACT

Evolutionary spectral theory provides a means for defining a decomposition of signal energy jointly over time and frequency for processes which exhibit an oscillatory behavior. The oscillatory model of behavior effectively assumes that a process is composed sinusoidal components with slowly-varying time-dependent amplitudes. In this paper we expand evolutionary spectral theory by allowing the frequency of the sinusoidal components to vary with time. An estimator of this generalized spectrum is described and examples are presented that illustrate the relative merits of this new approach.

1. INTRODUCTION

Since most signals display some form of non-stationarity, time-dependent spectra have been employed to describe the behavior of signals. Traditionally, non-stationary spectral definitions result from generalizations of wide sense stationary spectral theory. The short time Fourier transform and Bilinear Distributions (BD) [1] are examples of this generalization.

A second approach to defining a time-dependent spectrum is presented by Priestley [2]. Priestley shows how processes composed of slowly varying amplitude modulated carriers, termed oscillatory processes, can possess an evolutionary spectrum. We refer to Priestley's spectrum as the Oscillatory Evolutionary Spectrum (OES). Recently, Priestley's work has been expanded to include a dual definition and models that assume linearly varying Frequency Modulation(FM)[3].

In many cases the assumption that a signal is composed of amplitude modulated carriers is useful, however, not all signals are best characterized by this assumption. Consider FM signals. One would expect that such signals are better described in terms of sinusoids with varying frequency. Motivated by the desire to better characterize FM signals through the evolutionary spectral theory, we define the variable frequency or generalized evolutionary spectrum by applying the concept of Instantaneous Frequency (IF).

Cohen provides a detailed chronology of the development of concept of IF[1]. IF is defined as the rate of change of phase of the analytic signal. Where the analytic signal is obtained from the original signal by subtracting off the Hilbert transform of the signal.

The goal in defining IF is to obtain from signals involving modulation of the frequency of a sinusoid an expression that indicates the frequency of the sinusoid at every instance. However, only under certain conditions does the preceding definition provide a value that is physically meaningful[1, 4, 5]. IF defined as the derivative of the phase of the signal does provide a value that is physically meaningful for the so called pure FM signals[5], an example of which being

$$x(t) = Ae^{j\psi(t)t}, \tag{1}$$

where A is a constant and ψ is real valued. Later we employ the pure FM signal with the understanding that the derivative of the phase corresponds to the frequency of the signal at each instance.

In evolutionary spectral theory, many time domain processes admit the representation,

$$x(t) = \int \phi(t,\gamma)dZ(\gamma) \tag{2}$$

where $\{Z(\gamma)\}$ is an orthogonal increments process. Priestley showed that when the process is oscillatory, that is, a collection of components that are highly localized in frequency, the representation becomes

$$x(t) = \int A(t,\omega)e^{j\omega t}dZ(\omega) \tag{3}$$

and the spectral definition follows from the power of each sinusoid as

$$S_{OES}(t,\omega) = |A(t,\omega)|^2. \tag{4}$$

We next describe a generalization of this spectrum.

2. GENERAL OSCILLATORY MODEL

In this section we introduce a model of process behavior which generalizes the oscillatory model. With the oscillatory model, processes are assumed to be composed of phasors with time-dependent amplitudes and where the radian frequency of these phasors remains constant throughout time. The aspect of the oscillatory model that is generalized by this new model is the behavior of the phasor. Under the generalized model, processes are described in terms of phasors with slowly varying amplitude and an instantaneous frequency that varies with time in an arbitrary fashion. To construct such models, we first discuss the transformation of

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a deterministic signal and show how to construct a transformation that decomposes a signal into phasors whose instantaneous frequency varies in a desired way. Next we argue that allowing slow amplitude variations of the phasors leads to a valuable time-frequency description of a signal.

2.1. A Generalized Fourier Transformation

As our goal is to generalize the behavior of the time-dependent phasor, we will write the transformation of a deterministic signal in terms of phasors in as general a fashion as possible. The most general form for the transformation of $x(t)$ in terms of time dependent phasors is:

$$X(\lambda;\psi) = \int_{-\infty}^{\infty} x(t)e^{-j\psi(t,\lambda)}dt, \qquad (5)$$

where ψ is some real valued function. The notation $X(\lambda;\psi)$ is intended to emphasize that X depends explicitly on ψ. As we consider different forms of the function ψ, this notation will be adapted.

We now ask the question; under what conditions is (5) invertible? Clearly, this is a desirable property since $x(t)$ can be recovered from $X(\lambda;\psi)$ only if the transformation is invertible. The transformation of (5) is invertible if ψ is of the form:

$$\psi(t,\lambda) = \lambda t + p(t) + q(\lambda), \qquad (6)$$

where p and q are real valued functions. We now show that the inverse of (5) is then given by:

$$\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda;\psi)e^{j\psi(t,\lambda)}d\lambda. \qquad (7)$$

To see that ψ in the form of (6) is sufficient for invertibility, substitute (5) into (7):

$$\tilde{x}(t) = \frac{1}{2\pi}\iint x(\tau)e^{j[\lambda\tau+p(\tau)+q(\lambda)-\lambda t-p(t)-q(\lambda)]}d\tau d\lambda. \quad (8)$$

From here, we see that q cancels out and after changing the order of integration we find that

$$\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\tau)e^{-j[p(t)-p(\tau)]}\int_{-\infty}^{\infty} e^{-j\lambda[t-\tau]}d\lambda d\tau. \quad (9)$$

Performing the inner integration yields the desired result:

$$\begin{aligned}\tilde{x}(t) &= \int_{-\infty}^{\infty} x(\tau)e^{-j[p(t)-p(\tau)]}\delta(t-\tau)d\tau \\ &= x(t). \end{aligned} \qquad (10)$$

We now argue that the transformation of (5) is invertible in general only if ψ is of the form of (6). Consider the case where p were to depend on λ. In this case, the phasors involving p would not come out of the inner integral of (9) and hence the transformation would not in general be invertible. Similarly, if q were to depend on t, then the phasors involving q would not cancel in going from (8) to (9) and again the transformation would not in general be invertible.

As a result, the transformation pair that will be considered here is

$$X(\lambda;p,q) = \int_{-\infty}^{\infty} x(t)e^{-j[\lambda t+p(t)+q(\lambda)]}dt, \qquad (11)$$

with inverse

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda;p,q)e^{j[\lambda t+p(t)+q(\lambda)]}d\lambda \qquad (12)$$

The goal of this discussion is to investigate the role that the product λt and the functions p and q play in time-frequency analysis. Suppose that

$$X(\lambda;p,q) = 2\pi\delta(\lambda-\lambda_0), \qquad (13)$$

then

$$x(t) = e^{j[\lambda_0 t+p(t)+q(\lambda_0)]} \qquad (14)$$

or equivalently, $x(t)$ is a time-dependent phasor with instantaneous frequency:

$$\omega = \mathrm{IF}(t) = \lambda_0 + p'(t) \qquad (15)$$

where the prime indicates differentiation with respect to the argument. In light of the earlier discussions on IF, by specifying p we are able to control the trace of a basic element in the time-frequency plane.

To understand the role of q in this transformation, consider the same example as above; It is clear from (14) that q adds a time-invariant phase component. Since the spectrum is ultimately defined in terms of the magnitude square of the envelope, see (21), adding a constant phase component to the envelope will have no affect on the spectrum.

Finally, to understand the role of the product λt, consider once again the example from above. If λ_0 were allowed to vary in (14), the instantaneous frequency of the basic element would shift in frequency. That is, the λt product shifts a basic element in frequency.

Notice that if $\psi(t,\lambda) = \lambda t$, then (5) reduces to the Fourier transform. We refer to the transformation

$$X(\lambda;p) = \int_{-\infty}^{\infty} x(t)e^{-j[\lambda t+p(t)]}dt \qquad (16)$$

as the Generalized Fourier Transform (GFT), the inverse of which is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda;p)e^{j[\lambda t+p(t)]}d\lambda. \qquad (17)$$

In the next section, this transformation is used as a basis for obtaining a generalized spectral definition.

2.2. The Generalized Oscillatory Evolutionary Spectrum

We next motivate the generalized oscillatory model by means of an example. To develop this example we first consider the deterministic signal of equation (14). Intuitively, we expect that the time-frequency description of the frequency modulated signal of (14) is zero everywhere except along the line of the instantaneous frequency where the time-frequency description remains constant [6]. We next consider the effect of multiplying this signal by a window. If the window varies slowly in time, then a desirable result of a spectral definition is that for the windowed example, the amplitude of the spectrum is proportional to the amplitude squared of the window. We now consider modifications to the evolutionary spectral theory which will allow a spectral definition that possesses such a property. The spectral model just described is similar in spirit to Cohen's work on defining BDs that are concentrated along arbitrary lines in the time-frequency plane[6].

In defining this evolutionary spectrum we assume that the non-stationary process admits the representation of (2). The difficulty with defining a time-frequency decomposition from this representation lies in the physical meaning of the variable γ. In the case of the oscillatory model Priestley described conditions on the family of functions which ensured that this dependent variable has the physical interpretation of frequency. We now describe conditions on the family of functions which ensure that this dependent variable has the physical interpretation of an instantaneous frequency shift such as that of λ_0 in (14).

We proceed by expressing the family of functions, $\{\phi(t,\gamma)\}$, in a fashion which is consistent with the GFT. Throughout this discussion it is assumed that p of (16) is an arbitrary yet fixed function of time. We write

$$\phi(t,\gamma) = C(t,\gamma)e^{j[\lambda(\gamma)t+p(t)]} \tag{18}$$

where the function $\lambda(\gamma)$ is selected so that the magnitude of the Fourier transform of the envelope, $C(t,\gamma)$, with respect to t, assumed to exist, has a maximum at the zero frequency. The goal in making this assignment is to ensure that the envelope is dominated by the DC component and hence $\lambda(\gamma)$ may be interpreted as shifts in frequency.

If $\lambda(\gamma)$ is a singular valued function of γ, then a suitable variable substitution can be performed thereby enabling us to write

$$x(t) = \int_{-\infty}^{\infty} C(t,\lambda)e^{j[\lambda t+p(t)]}dZ(\lambda) \tag{19}$$

We obtain the spectral definition as an obvious consequence of the energy density of the process,

$$E\{|x(t)|^2\} = \int_{-\infty}^{\infty} |C(t,\lambda)|^2\, d\lambda. \tag{20}$$

This expression provides a decomposition of signal energy which after appropriate transformation can be interpreted in time and frequency. The necessary transformation follows from the instantaneous frequency of a signal component, (15). The Generalized Oscillatory Evolutionary Spectrum (GOES) defined with respect to the family of functions $\phi(t,\lambda) = C(t,\lambda)e^{j[\lambda t+p(t)]}$ is given by

$$S_{\mathrm{GOES}}(t,\omega;p) = \left|C(t,\omega - p'(t))\right|^2. \tag{21}$$

Observe that if $p(t)$ is constant, then the GOES reduces to the OES. Below we describe a means of estimating this spectrum.

3. SPECTRAL ESTIMATION

As a result of the of the similarity in the derivation of the OES and the GOES, estimators of the OES can be adapted to serve as estimators of the GOES. We consider modifications to the evolutionary periodogram estimator of the OES[7]. The Generalize Oscillatory Evolutionary Periodogram (GOEP) estimate of the time-dependent envelope is given by:

$$\hat{C}(t,\lambda;p) = \sum_{l=0}^{M-1} \beta_l^*(n) \sum_{m=0}^{N-1} \beta_l(m)x(m)e^{-j[\frac{2\pi}{N}\lambda n+p(n)]} \tag{22}$$

where $\{\beta_l(n)|l = 1,\ldots,M; n = 0,\ldots,N\}$ is a set of M orthonormal basis functions on $n = 0,\ldots,N$ which capture the time-dependent variations of the envelope $C(t,\lambda)$. The spectral estimate follows from the spectral definition as

$$\hat{S}_{\mathrm{GOEP}}(t,\omega;p) = \frac{N}{M}|\hat{C}(t,\omega - p'(t))|^2. \tag{23}$$

where the multiplicative constant $\frac{N}{M}$ was introduced by Kayhan to provide appropriate normalization.

To visualize the estimation process, consider that the inner summation of the estimator of (22) can be thought of as a projection of the complex demodulated signal onto the basis functions. The outer summation recombines the basis functions using the results of the earlier projections.

4. EXAMPLES

For this example we employ three signals, a sinusoid, a linear FM and a sinusoidal FM. These signals are intended to illustrate the relative merits of a variety of models and are respectively given by:

$$\begin{aligned} x_1(n) &= e^{j\pi n} \qquad \text{for} \qquad n = 0,\ldots,127 \\ x_2(n) &= e^{j[\frac{\pi n^2}{256}+\frac{\pi n}{2}]} \\ x_3(n) &= e^{j[32\cos(\frac{5\pi n}{256})+\pi n]} \end{aligned} \tag{24}$$

In Figure 1 we provide the time-dependent spectral estimates of the three example signals under three spectral models. In each case nine Fourier basis functions were employed and the value of p for the various models is given in the figure. From these examples, we conclude that each signal can be thought of as possessing slowly varying amplitude if the model is selected properly. Or in other words, the content of the signal determines which spectral model will provide the best estimate.

5. CONCLUSION

Evolutionary spectral theory is not limited to processes with slowly varying time-dependent frequency content. We have shown that by generalizing the spectral model to allow the possibility of varying the frequency of Priestley's sinusoid, the class of processes that evolutionary spectral theory can successfully be applied to increases significantly.

In defining the generalized spectrum, we allow for an infinity of spectral models, each of which is best suited for a specific signal type. The act of defining an infinity of spectral models suggests that, like stationary spectral theory, the spectral model for a non-stationary process should depend on the signal.

We have provided an estimator for this generalized spectrum and demonstrated using examples the suitability of a selection of models to a variety of signals.

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