TIME-FREQUENCY FORMULATION AND DESIGN OF NONSTATIONARY WIENER FILTERS*

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Abstract—The *nonstationary Wiener filter* (WF) is the optimum linear system for estimating a nonstationary signal contaminated by nonstationary noise. We propose a *time-frequency (TF) formulation* of nonstationary WFs for the practically important case of *underspread* processes. This TF formulation extends the spectral representation of stationary WFs to the nonstationary case, and it allows an approximate *TF design* of nonstationary WFs. For underspread processes, the performance obtained with the approximate TF design is close to that of the exact WF.

1 INTRODUCTION

We consider the following estimation problem [1]: A nonstationary, zero-mean random signal $s(t)$ with known autocorrelation function $r_s(t,t') = \mathcal{E}[s(t)s^*(t')]$ is contaminated by nonstationary, zero-mean, additive noise $n(t)$ with known autocorrelation function $r_n(t,t') = \mathcal{E}[n(t)n^*(t')]$ ($\mathcal{E}[\cdot]$ denotes the expectation operator). The signal and noise processes are uncorrelated, $\mathcal{E}[s(t)n^*(t')] \equiv 0$. Based on the observation $r(t) = s(t) + n(t)$, we form an estimate $\hat{s}(t)$ of the signal $s(t)$ using a linear, generally time-varying (LTV) system $\mathbf{H}$ with impulse response $h(t,t')$,[1]

$$\hat{s}(t) = (\mathbf{H}\,r)(t) = \int_{t'} h(t,t')\,r(t')\,dt' .$$

The optimum filter $\mathbf{H}_o$ minimizing the mean-square error $\mathcal{E}[|e(t)|^2]$, where $e(t) = s(t) - \hat{s}(t)$, is known as the (nonstationary) *Wiener filter* (WF),

$$\mathbf{H}_o \stackrel{\text{def}}{=} \operatorname*{argmin}_{\mathbf{H}} \mathcal{E}[|e(t)|^2] \qquad \forall\ t.$$

It is shown in [1] that $\mathbf{H}_o$ is a solution to the equation

$$\mathbf{H}_o(\mathbf{R}_s + \mathbf{R}_n) = \mathbf{R}_s \tag{1}$$

where $\mathbf{R}_s$ and $\mathbf{R}_n$ are the autocorrelation operators of $s(t)$ and $n(t)$, respectively, i.e., the linear operators whose kernels are the known autocorrelation functions $r_s(t,t')$ and $r_n(t,t')$, respectively. Furthermore, the autocorrelation operators of the signal estimate $\hat{s}_o(t)$ and estimation error $e_o(t) = s(t) - \hat{s}_o(t)$ obtained with the WF $\mathbf{H}_o$ are given by

$$\mathbf{R}_{\hat{s}_o} = \mathbf{H}_o\mathbf{R}_s, \qquad \mathbf{R}_{e_o} = \mathbf{H}_o\mathbf{R}_n. \tag{2}$$

Stationary processes. For wide-sense stationary processes $s(t)$ and $n(t)$ with power spectral densities $P_s(f)$ and $P_n(f)$, respectively, the WF $\mathbf{H}_o$ is a linear *time-invariant* system [1, 2]. It follows from (1) that the frequency response $H_o(f)$ of this system is given by

$$H_o(f) = \frac{P_s(f)}{P_s(f) + P_n(f)} \tag{3}$$

(assuming existence of this expression). With (2), the signal estimate $\hat{s}_o(t)$ and the estimation error $e_o(t)$ are stationary processes with power spectral densities

$$P_{\hat{s}_o}(f) = H_o(f)\,P_s(f) = \frac{P_s^2(f)}{P_s(f) + P_n(f)}, \tag{4}$$

$$P_{e_o}(f) = H_o(f)\,P_n(f) = \frac{P_s(f)\,P_n(f)}{P_s(f) + P_n(f)}. \tag{5}$$

This *frequency-domain formulation* allows an intuitively pleasing interpretation and a simple *frequency-domain design* of stationary WFs.

Outline of paper. This paper extends the frequency-domain formulation and design of stationary WFs to the practically important class of "underspread" nonstationary processes [3]. After a discussion of nonstationary WFs in Section 2, Section 3 proposes a *time-frequency (TF) formulation* of nonstationary WFs which is based on the Wigner-Ville spectrum of nonstationary processes [4, 5] and the Weyl symbol of linear operators [6]-[8]. In Section 4, this TF formulation is used to develop an approximate *TF design* of nonstationary WFs. Computer simulations show that, for underspread processes, the performance of the approximate WF is close to that of the exact WF.

2 NONSTATIONARY WIENER FILTERS

We first discuss some properties of the nonstationary WF defined by (1). Since $s(t)$ and $n(t)$ are uncorrelated, the autocorrelation operator of the observation $r(t) = s(t) + n(t)$ equals $\mathbf{R}_r = \mathbf{R}_s + \mathbf{R}_n$. Hence, (1) can be rewritten as $\mathbf{H}_o\mathbf{R}_r = \mathbf{R}_s$. The operator $\mathbf{R}_r$ is self-adjoint and positive semidefinite [1]. Let us define the *observation space* $\mathcal{S}_r$ as the range $\mathcal{R}[\mathbf{R}_r]$ of $\mathbf{R}_r$ [9]. $\mathcal{S}_r$ is spanned by all eigenfunctions $u_k(t)$ of $\mathbf{R}_r$ corresponding to positive (i.e. nonzero) eigenvalues λ_k. Let N_r denote the dimension of $\mathcal{S}_r$ (i.e., the number of positive eigenvalues, which may be infinite). From the Karhunen-Loève expansion [1] $r(t) = \sum_{k=1}^{N_r} r_k\,u_k(t)$ with $\mathcal{E}[|r_k|^2] = \lambda_k > 0$, it follows that the realizations of $r(t)$ are elements of the observation space $\mathcal{S}_r$. We note that $\mathcal{S}_r = \mathcal{S}_s \cup \mathcal{S}_n$ with the *signal space* $\mathcal{S}_s = \mathcal{R}[\mathbf{R}_s]$ and the *noise space* $\mathcal{S}_n = \mathcal{R}[\mathbf{R}_n]$.

Full-rank or rank-deficient process. We call the observation $r(t)$ a *full-rank process* if $\mathcal{S}_r = \mathcal{L}_2(\mathbb{R})$ where

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[1]Integrals go from $-\infty$ to $+\infty$.

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$\mathcal{L}_2(\mathbb{R})$ denotes the space of all square-integrable (finite-energy) signals. Here, the inverse of $\mathbf{R}_r$ exists, and the WF is obtained by solving $\mathbf{H}_o\mathbf{R}_r = \mathbf{R}_s$ as

$$\mathbf{H}_o = \mathbf{R}_s\mathbf{R}_r^{-1} = \mathbf{R}_s(\mathbf{R}_s + \mathbf{R}_n)^{-1}.$$

If, on the other hand, $r(t)$ is *rank-deficient* in the sense that $\mathcal{S}_r \subset \mathcal{L}_2(\mathbb{R})$, then the inverse of $\mathbf{R}_r$ does not exist, and the WF as defined by $\mathbf{H}_o\mathbf{R}_r = \mathbf{R}_s$ is *ambiguous*. We can partition the total signal space $\mathcal{L}_2(\mathbb{R})$ as $\mathcal{L}_2(\mathbb{R}) = \mathcal{S}_r \cup \mathcal{S}_r^{\perp}$, where $\mathcal{S}_r^{\perp}$ is the orthogonal complement space of $\mathcal{S}_r$ [9]. A special solution of $\mathbf{H}_o\mathbf{R}_r = \mathbf{R}_s$ is the "minimal WF" $\mathbf{H}_o^{(\min)}$ which suppresses all signals that are orthogonal on the observation space $\mathcal{S}_r$,

$$\left(\mathbf{H}_o^{(\min)}x\right)(t) \equiv 0 \qquad \text{for all } x(t) \in \mathcal{S}_r^{\perp}.$$

The minimal WF is given by

$$\mathbf{H}_o^{(\min)} = \mathbf{R}_s\mathbf{R}_r^{\#} = \mathbf{R}_s(\mathbf{R}_s + \mathbf{R}_n)^{\#}, \qquad (6)$$

where $\mathbf{R}_r^{\#} = (\mathbf{R}_s + \mathbf{R}_n)^{\#}$ is the pseudo-inverse of $\mathbf{R}_r$ [9]. From the minimal WF, all other solutions of $\mathbf{H}_o\mathbf{R}_r = \mathbf{R}_s$ can be derived as $\mathbf{H}_o = \mathbf{H}_o^{(\min)} + \mathbf{X}\mathbf{P}_r^{\perp}$, where $\mathbf{X}$ is an arbitrary linear operator and $\mathbf{P}_r^{\perp}$ denotes the orthogonal projection operator on the complement space $\mathcal{S}_r^{\perp}$. Different WFs $\mathbf{H}_o$ are identical for input signals $x(t) \in \mathcal{S}_r$ but in general different for $x(t) \notin \mathcal{S}_r$; this difference, however, does not influence the mean-square error since $r(t) \in \mathcal{S}_r$. Indeed, the autocorrelation operators of the signal estimate and error signal obtained with $\mathbf{H}_o$ can be shown to be

$$\mathbf{R}_{\hat{s}_o} = \mathbf{H}_o\mathbf{R}_s = \mathbf{H}_o^{(\min)}\mathbf{R}_s = \mathbf{R}_s(\mathbf{R}_s + \mathbf{R}_n)^{\#}\,\mathbf{R}_s, \qquad (7)$$

$$\mathbf{R}_{e_o} = \mathbf{H}_o\mathbf{R}_n = \mathbf{H}_o^{(\min)}\mathbf{R}_n = \mathbf{R}_s(\mathbf{R}_s + \mathbf{R}_n)^{\#}\,\mathbf{R}_n; \qquad (8)$$

hence, they equal the autocorrelation operators obtained with $\mathbf{H}_o^{(\min)}$. It can furthermore be shown that the signal estimate obtained with the general WF $\mathbf{H}_o$ is an element of the signal space, while the error signal is an element of the intersection of the signal and noise spaces:

$$\hat{s}_o(t) \in \mathcal{S}_s, \qquad e_o(t) = s(t) - \hat{s}_o(t) \in \mathcal{S}_s \cap \mathcal{S}_n. \qquad (9)$$

These results will now be reformulated in the TF plane in an intuitively appealing manner.

3 TIME-FREQUENCY FORMULATION

Weyl symbol and spreading function. Our TF formulation of nonstationary WFs will be based on a TF representation ("time-varying frequency response") of linear operators (LTV systems) known as the *Weyl symbol* (WS). The WS of a linear operator $\mathbf{H}$ is defined as [6]-[8]

$$L_{\mathbf{H}}(t,f) = \int_{\tau} h\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau \qquad (10)$$

where $h(t,t')$ is the impulse response (kernel) of $\mathbf{H}$. The Fourier transform of the WS is the *spreading function* [10]

$$\begin{aligned} S_{\mathbf{H}}(\tau,\nu) &= \mathcal{F}_{t\to\nu}\,\mathcal{F}_{f\to -\tau}\{L_{\mathbf{H}}(t,f)\} \\ &= \int_t h\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi\nu t}\, dt, \qquad (11) \end{aligned}$$

which describes the TF displacement effects caused by the operator (or LTV system) $\mathbf{H}$ [10].

If $\mathbf{H} = \mathbf{R}_x$ is the autocorrelation operator of a nonstationary random process $x(t)$, then the WS becomes the *Wigner-Ville spectrum* (WVS) of $x(t)$,

$$\overline{W}_x(t,f) = L_{\mathbf{R}_x}(t,f) = \int_{\tau} r_x\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau.$$

The WVS describes the average TF energy distribution of $x(t)$ [4, 5]. Furthermore, the spreading function of $\mathbf{R}_x$ is the *expected ambiguity function* (EAF) of $x(t)$,

$$\begin{aligned} \overline{A}_x(\tau,\nu) &= S_{\mathbf{R}_x}(\tau,\nu) = \mathcal{F}_{t\to\nu}\,\mathcal{F}_{f\to -\tau}\{\overline{W}_x(t,f)\} \qquad (12) \\ &= \int_t r_x\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi\nu t}\, dt, \qquad (13) \end{aligned}$$

which can be interpreted as a TF correlation function [3].

Underspread processes. A nonstationary process $x(t)$ is called *underspread* if its EAF $\overline{A}_x(\tau,\nu)$ is effectively restricted to a *small* rectangular region $\mathcal{A}_x$ about the origin of the (τ,ν)-plane [3], i.e., $\overline{A}_x(\tau,\nu) \approx 0$ for $(\tau,\nu) \notin \mathcal{A}_x$. In the case of an underspread process, process components far apart in the TF plane are (nearly) uncorrelated [3]. With (12), it is seen that the WVS of an underspread process is a 2-D lowpass function, i.e., a smooth function.

Two processes are called *jointly underspread* if their EAF supports are restricted to the *same* small region about the origin of the (τ,ν)-plane. Note that two underspread processes need not be jointly underspread.

TF formulation of the WF. Using (11) and (13), the operator equation $\mathbf{H}_o\mathbf{R}_r = \mathbf{R}_s$ defining the WF can be rewritten in terms of the operators' spreading functions as

$$\int_{\tilde{\tau}}\int_{\tilde{\nu}} S_{\mathbf{H}_o}(\tau-\tilde{\tau},\nu-\tilde{\nu})\,\overline{A}_r(\tilde{\tau},\tilde{\nu})\,e^{j\pi(\tau\tilde{\nu}-\nu\tilde{\tau})}d\tilde{\tau}d\tilde{\nu} = \overline{A}_s(\tau,\nu). \qquad (14)$$

If $s(t)$ and $n(t)$ are jointly underspread with EAF support region $\mathcal{A} = \mathcal{A}_s = \mathcal{A}_n$, then it follows with $\overline{A}_r(\tau,\nu) = \overline{A}_s(\tau,\nu) + \overline{A}_n(\tau,\nu)$ that also $r(t)$ is underspread with EAF support $\mathcal{A}$. Let us assume that also the support of the WF's spreading function $S_{\mathbf{H}_o}(\tau,\nu)$ is restricted to $\mathcal{A}$ (such a restriction seems indeed to be valid for jointly underspread $s(t)$, $n(t)$). With these assumptions, both the integration variables $\tilde{\tau},\tilde{\nu}$ and the external variables τ,ν in (14) are restricted such that $\pi(\tau\tilde{\nu} - \nu\tilde{\tau}) \ll 1$ and thus $\exp\{j\pi(\tau\tilde{\nu} - \nu\tilde{\tau})\} \approx 1$. Hence, (14) can be approximately replaced by the convolution

$$\int_{\tilde{\tau}}\int_{\tilde{\nu}} S_{\mathbf{H}_o}(\tau-\tilde{\tau},\nu-\tilde{\nu})\,\overline{A}_r(\tilde{\tau},\tilde{\nu})\,d\tilde{\tau}d\tilde{\nu} \approx \overline{A}_s(\tau,\nu). \qquad (15)$$

Taking the Fourier transform of (15) yields

$$L_{\mathbf{H}_o}(t,f)\,\overline{W}_r(t,f) \approx \overline{W}_s(t,f), \qquad (16)$$

which is the TF formulation of the operator relation $\mathbf{H}_o\mathbf{R}_r = \mathbf{R}_s$. This equation shows that the WS $L_{\mathbf{H}_o}(t,f)$ of the WF is *ambiguous* for all TF points where $\overline{W}_r(t,f) = 0$. This is analogous to the ambiguity of $\mathbf{H}_o$ on the complementary observation space $\mathcal{S}_r^{\perp}$ (cf. Section 2). Let $\mathcal{R}_r$ denote the effective TF support of the observation $r(t)$ (i.e., $\overline{W}_r(t,f) \approx 0$ for $(t,f) \notin \mathcal{R}_r$). A "minimal" solution of (16), corresponding to the minimal WF in (6), is obtained by setting $L_{\mathbf{H}_o}(t,f) = 0$ for $(t,f) \notin \mathcal{R}_r$ (i.e., in the TF region where $r(t)$ does not have any energy). This yields

$$L_{\mathbf{H}_o}(t,f) \approx \begin{cases} \dfrac{\overline{W}_s(t,f)}{\overline{W}_s(t,f) + \overline{W}_n(t,f)}, & (t,f) \in \mathcal{R}_r \\ 0, & (t,f) \notin \mathcal{R}_r, \end{cases} \qquad (17)$$

where $\overline{W}_r(t,f) = \overline{W}_s(t,f) + \overline{W}_n(t,f)$ has been used. A similar derivation based on (2) shows that the WVS of the signal estimate $\hat{s}_o(t)$ is approximately given by $\overline{W}_{\hat{s}_o}(t,f) \approx L_{\mathbf{H}_o}(t,f)\,\overline{W}_s(t,f)$ (cf. (7)) and further by

$$\overline{W}_{\hat{s}_o}(t,f) \approx \begin{cases} \dfrac{\left[\overline{W}_s(t,f)\right]^2}{\overline{W}_s(t,f) + \overline{W}_n(t,f)}, & (t,f) \in \mathcal{R}_r \\ 0, & (t,f) \notin \mathcal{R}_r. \end{cases} \tag{18}$$

Similarly, the WVS of the estimation error $e_o(t)$ is $\overline{W}_{e_o}(t,f) \approx L_{\mathbf{H}_o}(t,f)\,\overline{W}_n(t,f)$ (cf. (8)) and further

$$\overline{W}_{e_o}(t,f) \approx \begin{cases} \dfrac{\overline{W}_s(t,f)\,\overline{W}_n(t,f)}{\overline{W}_s(t,f) + \overline{W}_n(t,f)}, & (t,f) \in \mathcal{R}_r \\ 0, & (t,f) \notin \mathcal{R}_r. \end{cases} \tag{19}$$

The WVS relations (18) and (19) (which hold for the general WF) are the TF formulation of the operator relations (7) and (8), respectively.

Interpretation. Eqs. (17)-(19) are the desired extension of the simple frequency-domain expressions (3)-(5), respectively, to nonstationary WFs. For stationary processes, (17)-(19) duly reduce to (3)-(5). The TF formulation (17)-(19) allows a simple and intuitively pleasing interpretation of nonstationary WFs in the underspread case (see *Fig. 1*):

- In the "signal-only" TF region $\mathcal{R}_s \setminus \mathcal{R}_n$ where $\overline{W}_s(t,f) \neq 0$ and $\overline{W}_n(t,f) \approx 0$, we have $L_{\mathbf{H}_o}(t,f) \approx 1$, i.e., the WF passes the signal.
- In the "noise-only" TF region $\mathcal{R}_n \setminus \mathcal{R}_s$ where $\overline{W}_n(t,f) \neq 0$ and $\overline{W}_s(t,f) \approx 0$, we have $L_{\mathbf{H}_o}(t,f) \approx 0$, i.e., the WF suppresses the noise.
- In the "signal+noise" TF region $\mathcal{R}_s \cap \mathcal{R}_n$ where $\overline{W}_s(t,f) \neq 0$ and $\overline{W}_n(t,f) \neq 0$, $L_{\mathbf{H}_o}(t,f)$ is approximately between 0 and 1, depending on the relative local strengths of signal and noise. For example, $L_{\mathbf{H}_o}(t,f) \approx 1/2$ for (t,f) where $\overline{W}_s(t,f) \approx \overline{W}_n(t,f)$.
- The TF support of the WVS of the signal estimate, $\overline{W}_{\hat{s}_o}(t,f)$, is restricted to the TF support $\mathcal{R}_s$ of the signal (where $\overline{W}_s(t,f) \neq 0$).
- The TF support of the WVS of the estimation error, $\overline{W}_{e_o}(t,f)$, is restricted to the "signal+noise region" $\mathcal{R}_s \cap \mathcal{R}_n$ (where $\overline{W}_s(t,f) \neq 0$ and $\overline{W}_n(t,f) \neq 0$).

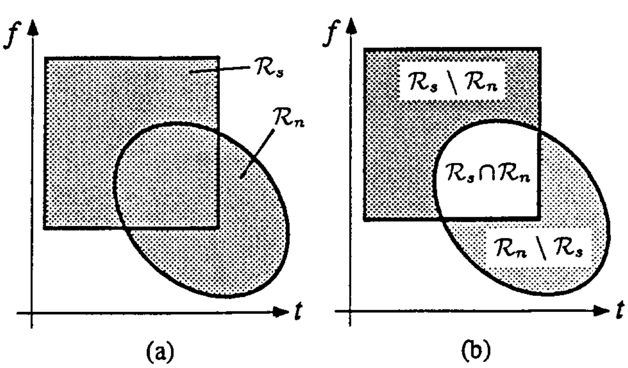

Fig. 1. (a) Effective TF supports of signal, $\mathcal{R}_s$, and noise, $\mathcal{R}_n$, (b) "signal-only" region $\mathcal{R}_s \setminus \mathcal{R}_n$, "noise-only" region $\mathcal{R}_n \setminus \mathcal{R}_s$, and "signal+noise" region $\mathcal{R}_s \cap \mathcal{R}_n$.

We note that the last two results are analogous to the relations (9).

Simulation results. The above TF formulation is verified experimentally in *Fig. 2*. The WVS of the signal and noise processes (generated by the TF synthesis method described in [11]) are shown in Figs. 2(a),(b). The WS of the WF (shown in Fig. 2(c)) and its approximation (17) (shown in Fig. 2(d)) are seen to be very similar. Furthermore, the WS of the WF is ≈ 1 in the signal-only region, ≈ 0 in the noise-only region, and $\approx 1/2$ where $\overline{W}_s(t,f) \approx \overline{W}_n(t,f)$ (cf. Fig. 1(b)). The WVS of the estimation error (shown in Fig. 2(e)) is in fact restricted to the "signal+noise region."

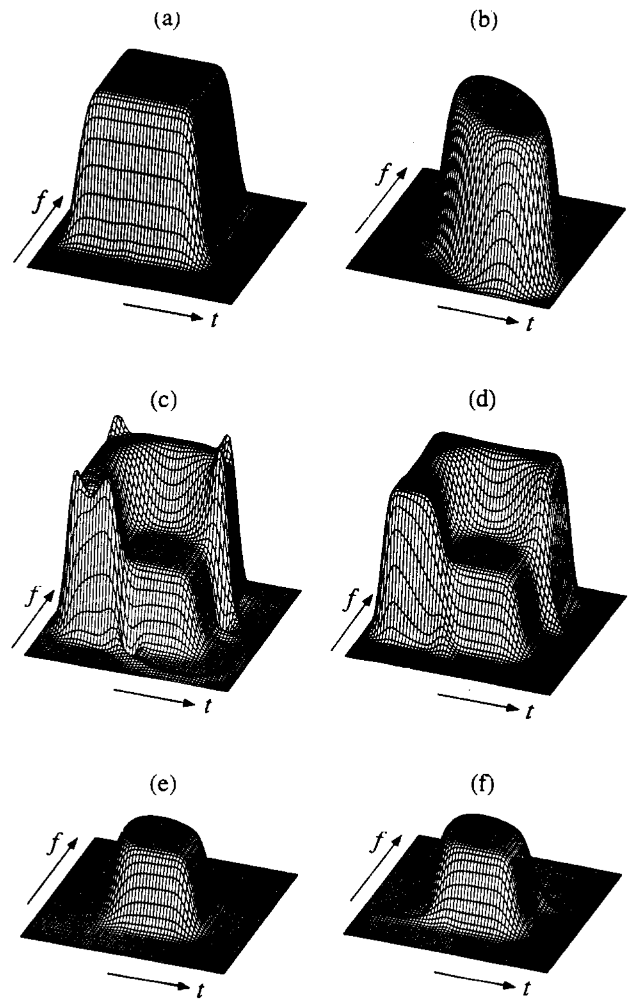

Fig. 2. TF analysis and TF design of nonstationary WFs: (a) WVS of signal $s(t)$, (b) WVS of noise $n(t)$, (c) WS (slightly smoothed) of WF $\mathbf{H}_o$, (d) approximation of WS of WF according to (17), simultaneously WS of Weyl filter $\mathbf{H}_1$, (e) WVS of error $e_o(t)$ of WF, and (f) WVS of error $e_1(t)$ of Weyl filter.

4 TIME-FREQUENCY DESIGN

The approximate expression (17) for the WS of the WF suggests a *TF design of nonstationary WFs*. Let us define the linear, time-varying filter $\mathbf{H}_1$ whose WS is equal to the right-hand side of (17),

$$L_{\mathbf{H}_1}(t,f) \stackrel{\text{def}}{=} \begin{cases} \dfrac{\overline{W}_s(t,f)}{\overline{W}_s(t,f)+\overline{W}_n(t,f)}, & (t,f)\in\mathcal{R}_r \\ 0, & (t,f)\notin\mathcal{R}_r. \end{cases} \quad (20)$$

We shall call $\mathbf{H}_1$ a *Weyl filter* since it is defined in terms of its WS [12]-[14]. The impulse response (kernel) $h_1(t,t')$ of the Weyl filter is easily obtained by inversion of (10),

$$h_1(t,t') = \int_f L_{\mathbf{H}_1}\Big(\frac{t+t'}{2},f\Big)e^{j2\pi(t-t')f}\,df, \quad (21)$$

and the signal estimate is finally calculated as

$$\hat{s}_1(t) = (\mathbf{H}_1 r)(t) = \int_{t'} h_1(t,t')\,r(t')\,dt'.$$

For jointly underspread $s(t)$ and $n(t)$ where (17) holds, we have $L_{\mathbf{H}_1}(t,f) \approx L_{\mathbf{H}_o}(t,f)$. It can furthermore be shown that $\overline{W}_{\hat{s}_1}(t,f) \approx \overline{W}_{\hat{s}_o}(t,f)$ and $\overline{W}_{e_1}(t,f) \approx \overline{W}_{e_o}(t,f)$, where $e_1(t)=s(t)-\hat{s}_1(t)$ denotes the error signal obtained with the Weyl filter. Hence, the Weyl filter $\mathbf{H}_1$ will closely approximate the WF $\mathbf{H}_o$. However, if $s(t)$, $n(t)$ are not jointly underspread, (17) is not valid, the Weyl filter will be widely different from the WF, and its performance will not be satisfactory.

The Weyl filter has two advantages over the WF:

- The *a priori* information required for its design is given by the WVS of signal and noise, and is thus formulated in the intuitively meaningful TF plane.
- The TF design of the Weyl filter is less expensive than that of the WF since the inversion of an operator (cf. (6)) is replaced by simple scalar inversions (cf. (20)) followed by Fourier transforms (cf. (21)).

Simulation results. Fig. 2(d) shows the WS of the Weyl filter for the signal and noise processes whose WVS are depicted in Figs. 2(a),(b). Note the close similarity to the WS of the WF in Fig. 2(c). The WVS of the error signal obtained with the Weyl filter, shown in Fig. 2(f), is nearly identical to that obtained with the WF (see Fig. 2(e)). The average SNR improvement[2] achieved by the WF is 5.08 dB, while that of the Weyl filter is 4.99 dB. Hence, the WF performs only slightly better than the Weyl filter. Realizations of the signal $s(t)$, noisy signal (observation) $r(t)$, signal estimates $\hat{s}_o(t)$ and $\hat{s}_1(t)$, and estimation error signals $e_o(t)$ and $e_1(t)$ are depicted in *Fig. 3*.

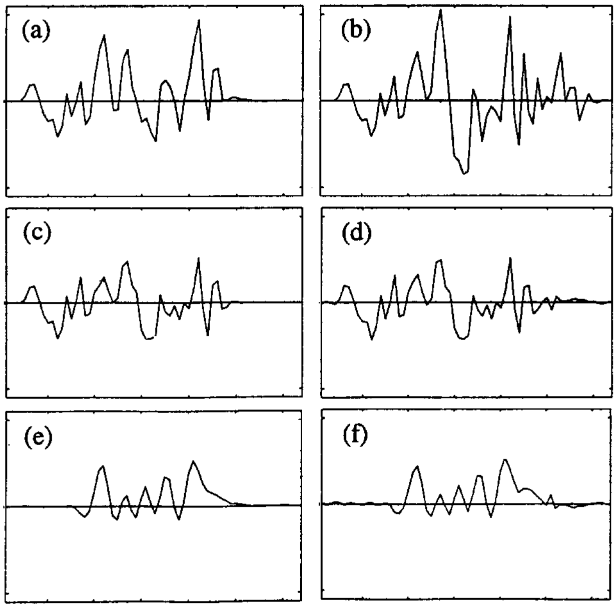

Fig. 3. Realizations of signal, observation, signal estimates, and error processes (real parts shown): (a) signal $s(t)$, (b) observation $r(t)$, (c) signal estimate $\hat{s}_o(t)$ of WF, (d) signal estimate $\hat{s}_1(t)$ of Weyl filter, (e) error $e_o(t)$ of WF, and (f) error $e_1(t)$ of Weyl filter.

5 CONCLUSIONS

We have extended the frequency-domain formulation, interpretation, and design of stationary WFs to a TF formulation, interpretation, and design of nonstationary WFs. This extension is based on the Weyl symbol and the Wigner-Ville spectrum, and is valid for nonstationary signal and noise processes that are jointly underspread. The performance of the approximate WF obtained by the proposed TF design is nearly as good as that of the exact WF.

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[2] The average SNR improvement is defined as $\mathrm{SNR_{in}}/\mathrm{SNR_{out}}$ where $\mathrm{SNR_{in}} = \mathcal{E}[\|s\|^2]/\mathcal{E}[\|n\|^2]$ and $\mathrm{SNR_{out}} = \mathcal{E}[\|s\|^2]/\mathcal{E}[\|e_o\|^2]$ (case of WF) or $\mathrm{SNR_{out}} = \mathcal{E}[\|s\|^2]/\mathcal{E}[\|e_1\|^2]$ (case of Weyl filter).