# $M$-BAND WAVELET DECOMPOSITION OF SECOND ORDER RANDOM PROCESSES

*J.-C. Pesquet* ‡

Laboratoire des Signaux et Systèmes, CNRS/UPS and GDR TdSI,
ESE, 91192 Gif sur Yvette Cédex, France,
e-mail : pesquet@lss.supelec.fr

## ABSTRACT

In this paper we investigate a $M$-band wavelet decomposition of second order random processes. In particular, we propose an extension of results which are known for the dyadic wavelet transform. The statistical properties of the $M$-band wavelet coefficients are listed and recursive relations are derived and used to compute their multiscale characteristics. Special attention is also paid to the multiscale analysis of linear parametric models.

## 1. Introduction

There has recently been a growing interest in the multiscale characterization of random processes [1, 2, 3, 4, 5, 6]. The objective of most of these works was to develop a new estimation/decision framework which would allow to integrate multiscale informations and reduce the computational complexity of conventional algorithms. A problem which is frequently encountered in this context is the ability to relate the statistical properties of a process often available at the highest resolution to those at low resolutions.

In this paper, we investigate the second order statistical properties of the $M$-band wavelet coefficients of a random process. In Section 2, the principles of the $M$-band wavelet decomposition are quickly reviewed and our notations are introduced. Section 3 gives the basic expressions of the correlations of a process in the time-scale domain. The exponential decay properties of the correlation sequences are outlined in Section 4. Section 5 provides scale recursive formula to compute the second order moments. Finally, we focus on the multiscale characterization of Auto-Regressive Integrated Moving Average (ARIMA) models, in Section 6.

‡ The author would like to thank H. Krim for his help and many fruitful discussions.

## 2. Notations

The $M$-band wavelet decomposition [7] ($M \in \mathbb{N} \setminus \{0\}$) generalizes the 2-band wavelet decomposition [8] which has gained much popularity in the recent years. Its definition involves a scaling function $\psi_0(t)$ and $M-1$ analyzing wavelets $(\psi_p(t))_{p\in\mathbb{Z}}$ in $\mathrm{L}^2(\mathbb{R})$. The $M$-band analysis of a second order complex process $\mathrm{x}(t)$ allows one to extract its features at scale $M^j$, $j \in \mathbb{Z}$, by calculating its wavelet (resp. approximation) coefficients, for $p \in \{1, \ldots, M-1\}$ (resp. $p = 0$),

$$c_j^p(k) \triangleq \int_{-\infty}^{\infty} \mathrm{x}(t)\, \frac{1}{M^{j/2}} \psi_p(\frac{t}{M^j} - k)^*\, dt, \quad k \in \mathbb{Z}\,. \quad (1)$$

The above integral exists in the mean square sense if

$$\int_{-\infty}^{\infty} \mathrm{E}\{|\, \mathrm{x}(t)\,|^2\}\, \frac{1}{M^{j/2}}\, |\, \psi_p(\frac{t}{M^j} - k)\,|\, dt < \infty\,. \quad (2)$$

To construct orthonormal wavelet bases, it is convenient to define multiresolution analyses of $\mathrm{L}^2(\mathbb{R})$. This requires equations relating the basis functions lying at two successive scales, *i.e.*

$$\frac{1}{\sqrt{M}} \psi_p(\frac{t}{M}) = \sum_{k=-\infty}^{\infty} h_p(-k)^*\, \psi_0(t-k), \quad 0 \le p < M, \quad (3)$$

where $(h_p(k))_{k\in\mathbb{Z}}$ is a sequence in $\ell^2(\mathbb{Z})$. This relation also leads to an efficient computation of the wavelet coefficients by cascading $M$-channel Quadrature Mirror Filter (QMF) banks, whose impulse responses are the sequences $(h_p(k))_{k\in\mathbb{Z}}$. When this filter bank is paraunitary, the frequency responses $H_p(\omega)$ of the filters are such that

$$\sum_{m=0}^{M-1} H_p(\frac{\omega + 2\pi m}{M})\, H_q(\frac{\omega+2\pi m}{M})^* = M\, \delta(p-q)\,, \quad (4)$$

which entails the orthonormality of the corresponding wavelet basis $(M^{-j/2}\, \psi_p(t/M^j - k))_{(k,j)\in\mathbb{Z}^2, 1\le p<M}$ of $\mathrm{L}^2(\mathbb{R})$. Another desirable property of the wavelets is



0-7803-2431-5/95 $4.00 © 1995 IEEE

their orthogonality to polynomials up to some order $r \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} t^k \, \psi_p(t) \, dt = 0, \quad 1 \leq p < M, \, 1 \leq k < r \,, \quad (5)$$

in order to ensure some "regularity" of these functions. This $r$-vanishing property is satisfied iff

$$H_p(\omega) = (1 - e^{i\omega})^r \, \widetilde{H}_p(\omega), \quad 1 \leq p < M \,, \quad (6)$$

where $\widetilde{H}_p(\omega)$ is locally bounded around 0.

We denote the cross-correlation of two (second order) complex random processes $\mathrm{x}(t)$ and $\mathrm{y}(t)$ by

$$\gamma_{\mathrm{xy}}(t, u) \triangleq \mathrm{E}\{\mathrm{x}(t)\mathrm{y}(u)^*\} \quad (7)$$

and their cross-spectrum density is defined as

$$\widehat{\gamma}(\omega, \nu) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{\mathrm{xy}}(t, u) \, e^{-i(\omega t - \nu u)} \, dt \, du \quad (8)$$

$$(\text{resp.}\, \widehat{\gamma}(\omega, \nu) \triangleq \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \gamma_{\mathrm{xy}}(k, l) \, e^{-i(\omega k - \nu l)}) \,,$$

for continuous (resp. discrete) time signals. In the stationary case, $\gamma_{\mathrm{xy}}(t, t - \tau) \triangleq R_{\mathrm{xy}}(\tau)$, whose Fourier transform is $\widehat{R}_{\mathrm{xy}}(\omega)$. When $\mathrm{y}(t) = \mathrm{x}(t)$, the above second order moments reduce to the autocorrelation $\gamma_{\mathrm{x}}(t, u)$ (or $R_{\mathrm{x}}(\tau)$) and the power spectrum density $\widehat{\gamma}_{\mathrm{x}}(\omega, \nu)$ (or $\widehat{R}_{\mathrm{x}}(\omega)$).

## 3. Second Order Statistics

The expressions concerning the second order statistics of the 2-band wavelet coefficients of a second order process [1] x(t) may be readily extended to the $M$-band case.

**Property 1** *The cross-correlations* $\gamma_{c_j^p c_{j+s}^q}(k, l)$, $(p, q) \in \{0, \ldots, M - 1\}^2$, $(j, s) \in \mathbb{Z}^2$, *are obtained from the autocorrelation* $\gamma_{\mathrm{x}}(t, u)$ *as follows:*

$$\gamma_{c_j^p c_{j+s}^q}(k, l) = M^{j+s/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{\mathrm{x}}(M^j(t + k), M^{j+s}(u + l)) \, \psi_p(t)^* \, \psi_q(u) \, dt \, du \,. \quad (9)$$

*We have equivalently in the frequency domain:*

$$\widehat{\gamma}_{c_j^p c_{j+s}^q}(\omega, \nu) = \frac{1}{M^{j+s/2}} \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \widehat{\gamma}_{\mathrm{x}}(\frac{\omega - 2\pi k}{M^j}, \frac{\nu - 2\pi l}{M^{j+s}}) \, \widehat{\psi}_p(\omega - 2\pi k)^* \, \widehat{\psi}_q(\nu - 2\pi l) \,. \quad (10)$$

The above results can be further simplified in the stationary case.

**Corollary 1** *When* $\mathrm{x}(t)$ *is a stationary process, for all* $s \geq 0$ *and* $m \in \{0, \ldots, M^s - 1\}$, $(c_{j,m}^{p,s}(k) \triangleq c_j^p(M^s k + m))_{k \in \mathbb{Z}}$ *and* $(c_{j+s}^q(k))_{k \in \mathbb{Z}}$ *are cross-stationary sequences such that*

$$R_{c_{j,m}^{p,s} c_{j+s}^q}(k) = M^j \int_{-\infty}^{\infty} R_{\mathrm{x}}(M^j(\tau + m + M^s k)) \, A_{\psi_p \psi_q}(s, \tau)^* \, d\tau \,, \quad (11)$$

*where* $A_{\psi_p \psi_q}(s, \tau)$ *is the wide band cross-ambiguity function defined by*

$$A_{\psi_p \psi_q}(s, \tau) \triangleq \frac{1}{M^{s/2}} \int_{-\infty}^{\infty} \psi_p(t) \, \psi_q(\frac{t - \tau}{M^s})^* \, dt \,. \quad (12)$$

*The cross-spectrum density of the two processes is then given by*

$$\widehat{R}_{c_{j,m}^{p,s} c_{j+s}^q}(\omega) = \frac{1}{M^{s/2}} \sum_{k=-\infty}^{\infty} \widehat{R}_{\mathrm{x}}(\frac{\omega - 2\pi k}{M^s}) \, e^{im\frac{\omega - 2\pi k}{M^s}} \, \widehat{\psi}_p(\frac{\omega - 2\pi k}{M^s})^* \, \widehat{\psi}_q(\omega - 2\pi k) \,. \quad (13)$$

It must be pointed out that the cross-stationarity of the sequences $(c_j^p(k))_{k \in \mathbb{Z}}$ and $(c_{j+s}^q(k))_{k \in \mathbb{Z}}$, resulting from the decomposition of a stationary random process, is only guaranteed if $s = 0$. Note also that the cross-stationarity of all the sequences $(c_{j,m}^{p,s}(k))_{k \in \mathbb{Z}}$ and $(c_{j+s}^q(k))_{k \in \mathbb{Z}}$, $s \geq 0$, $p > 0$, $q > 0$, is not generally a sufficient condition for the stationarity of the analyzed process. In particular, the correlated noise which is synthesized by setting $\gamma_{c_j^p c_{j+s}^q}(k, l) = \sigma_{j,p}^2 \, \delta(p - q) \, \delta(k - l) \, \delta(s)$, $p > 0$, $q > 0$, is not necessarily stationary.

## 4. Some Asymptotic Results

We now derive upper bounds on the correlations of $M$-band wavelet coefficients, which are useful to highlight their asymptotic behaviors. We will first state the results concerning intrascale correlations.

**Property 2** *Let* $\mathrm{x}(t)$ *be a stationary process whose autocorrelation is exponentially decaying,*

$$| \, R_{\mathrm{x}}(\tau) \, | \leq R_{\mathrm{x}}(0) \, e^{-\alpha |\tau|}, \quad \alpha \geq 0. \quad (14)$$

*If* $| \, A_{\psi_p \psi_q}(0, \tau) \, | \leq \bar{A}_{\psi_p \psi_q}(\tau)$, *for all* $\tau \in \mathbb{R}$ *where* $\bar{A}_{\psi_p \psi_q}(\tau)$ *is a real Lipschitz function such that, for all* $(\tau_1, \tau_2) \in \mathbb{R}^2$,

$$| \, \bar{A}_{\psi_p \psi_q}(\tau_1) - \bar{A}_{\psi_p \psi_q}(\tau_2) \, | \leq K_{pq} \, | \, \tau_1 - \tau_2 \, |, \quad K_{pq} \geq 0 \,, \quad (15)$$

*then*

$$| R_{c_j^p c_j^q}(k) | \leq \frac{2}{\alpha} R_x(0)(\bar{A}_{\psi_p\psi_q}(-k) + \frac{K_{pq}}{\alpha M^j}). \quad (16)$$

*Proof*: According to Eqs. (11) and (14), we have

$$| R_{c_j^p c_j^q}(k) | \leq M^j \, R_x(0) \int_{-\infty}^{\infty} e^{-\alpha M^j |\tau + k|} \, \bar{A}_{\psi_p\psi_q}(\tau) \, d\tau \, . \quad (17)$$

Furthermore, by using (15), we find that

$$\int_{-\infty}^{\infty} e^{-\alpha M^j |\tau+k|} \, | \bar{A}_{\psi_p\psi_q}(\tau) - \bar{A}_{\psi_p\psi_q}(-k) | \, d\tau$$

$$\leq K_{pq} \int_{-\infty}^{\infty} e^{-\alpha M^j |\tau|} \, | \tau | \, d\tau = \frac{2K_{pq}}{\alpha^2 M^{2j}} \quad (18)$$

■

This result means that the intrascale correlations of the wavelet coefficients exponentially decay w.r.t. $j$. It is important to note that Assumption (15) is fulfilled for two interesting kinds of wavelets:

- compactly supported wavelets, *i.e.*

$$\begin{aligned} | \psi_p(t) | &\leq \|\psi_p\|_\infty \mathrm{rect}_T(t) \, , \, T > 0 \, , & (19)\\ \bar{A}_{\psi_p\psi_q}(\tau) &= K_{pq}(T - | \tau |)\mathrm{rect}_{2T}(t) \, , & (20)\\ K_{pq} &= \|\psi_p\|_\infty \, \|\psi_q\|_\infty \, ; & (21) \end{aligned}$$

- exponentially decaying wavelets, *i.e.*

$$\begin{aligned} | \psi_p(t) | &\leq \|\psi_p\|_\infty e^{-\beta|t|} \, , \, \beta > 0 \, , & (22)\\ \bar{A}_{\psi_p\psi_q}(\tau) &= e \, K_{pq}(\frac{1}{\beta} + | \tau |) \, e^{-\beta|\tau|} \, , & (23)\\ K_{pq} &= \|\psi_p\|_\infty \, \|\psi_q\|_\infty / e \, . & (24) \end{aligned}$$

Some more accurate bounds may be found in these two cases by using the specific form of $\bar{A}_{\psi_p\psi_q}(\tau)$.

We now turn our attention to interscale bounds.

**Property 3** *Let* $x(t)$ *be a stationary process satisfying Eq. (14). If there exists* $a_{\psi_p\psi_q} \geq 0$ *such that, for all* $\tau \in \mathbf{R}$ *and* $s \geq 0$,

$$| A_{\psi_p\psi_q}(s, \tau) | \leq \frac{a_{\psi_p\psi_q}}{M^{s/2}} \, , \quad (25)$$

*then*

$$| R_{c_{j,m}^{p,s} c_{j+s}^q}(k) | \leq \frac{2a_{\psi_p\psi_q}}{\alpha M^{s/2}} R_x(0) \, . \quad (26)$$

This means that the interscale correlations exponentially decay w.r.t. $s$. As previously, compactly supported (resp. exponentially decaying) wavelets appear as special cases of wavelets satisfying Condition (25), with $a_{\psi_p\psi_q} = T \, \|\psi_p\|_\infty \, \|\psi_q\|_\infty$ (resp. $a_{\psi_p\psi_q} = 2/\beta \, \|\psi_p\|_\infty \, \|\psi_q\|_\infty$).

## 5. Scale Recursive Formulations

In practical situations, it is often convenient to determine the second order statistics of the $M$-band wavelet coefficients by making use of Eq. (3). Scale recursive relations are thus obtained to compute the intrascale correlations:

**Property 4** *The cross-correlations* $\gamma_{c_{j+1}^p c_{j+1}^q}(k, l)$, $(p, q) \in \{0, \ldots, M-1\}^2$, *are obtained from the autocorrelation* $\gamma_{c_j^0}(k, l)$ *by the 2D filter bank shown in Fig. 1. If the analyzed signal is stationary, this structure reduces to a 1D filter bank, as shown in Fig. 2.*

The interscale correlations $\gamma_{c_j^p c_{j+s}^q}(k, l)$ or $R_{c_{j,r}^{p,s} c_{j+s}^q}(k)$, $s \geq 0$, may be derived in a similar way. Furthermore, when an orthonormal wavelet basis is used, conservation laws of the autocorrelation sequences are deduced from the above property:

**Corollary 2** *The orthonormal wavelet decomposition of a stationary process is such that*

$$M \, R_{c_j^0}(Mk) = \sum_{p=0}^{M-1} R_{c_{j+1}^p}(k) \, , \quad (27)$$

$$M \, R_{c_j^0} * R_{c_j^0}(Mk)$$

$$= \sum_{p=0}^{M-1} \sum_{q=0}^{M-1} R_{c_{j+1}^q c_{j+1}^p} * R_{c_{j+1}^p c_{j+1}^q}(k) \, , \quad (28)$$

*wher "*" designates the convolution of two sequences.*

*Proof* : As a consequence of Eq. (4), we have

$$\sum_{m=0}^{M-1} \widehat{R}_{c_{j,m}^{0,1}}(\omega) = \sum_{p=0}^{M-1} \widehat{R}_{c_{j+1}^p}(\omega) \, , \quad (29)$$

$$\sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} | \widehat{R}_{c_{j,m}^{0,1} c_{j,m'}^{0,1}}(\omega) |^2 =$$

$$\sum_{p=0}^{M-1} \sum_{q=0}^{M-1} | \widehat{R}_{c_{j+1}^p c_{j+1}^q}(\omega) |^2 \, , \quad (30)$$

which lead to the desired relations.

## 6. Decomposition of Linear Parametric Models

It is often useful to model the approximation coefficients at the highest resolution $M^{-j_0}$ of a multiscale analysis by a discrete time ARMA process. This assumption is relatively weak as it is particularly valid if the redundant approximation coefficients

$$C_{j_0}^0(t) \triangleq \int_{-\infty}^{\infty} x(\theta) \, \frac{1}{M^{j_0/2}} \psi_0(\frac{\theta - t}{M^{j_0}})^* \, d\theta \quad (31)$$

can be modeled by a continuous time ARMA process. The following property can then be established:

**Property 5** *When $(c^0_{j_0}(k))_{k\in\mathbb{Z}}$ is an ARMA process and FIR multiresolution filters are used, the sequences $(c^p_j(k))_{k\in\mathbb{Z}}$, $j > j_0$, are also ARMA processes. The AR part $A^p_{j+1}(\omega)$ is equal to $\prod_{q=0}^{M-1} A^0_j(\frac{\omega+2\pi q}{M})$ and the MA part $B^p_{j+1}(\omega)$ is a polynomial in $e^{i\omega}$ which is obtained by a spectral factorization of $M^{-1}\sum_{q=0}^{M-1} \mid H_p(\frac{\omega+2\pi q}{M}) \mid^2 \mid B^0_j(\frac{\omega+2\pi q}{M}) \mid^2 \prod_{m=0,m\neq q}^{M-1} \mid A^0_j(\frac{\omega+2\pi m}{M}) \mid^2$.*

*Proof*: The expressions of $\mid A^p_{j+1}(\omega) \mid^2$ and $\mid B^p_{j+1}(\omega) \mid^2$ are straightforwardly obtained from the decimation rules. These functions are (non causal) polynomials in $e^{i\omega}$, since $\mid A^p_{j+1}(M\omega) \mid^2$ and $\mid B^p_{j+1}(M\omega) \mid^2$ are polynomials which are invariant through any shift of $\omega$ by $2\pi m/M$, $m \in \{1,\ldots,M-1\}$. ■

These results can be extended to ARIMA nonstationary processes and, as in the 2-band case [6]:

**Corollary 3** *Let an r-vanishing M-band wavelet decomposition be implemented by a QMF filter bank with FIR analysis filters of length $P+1$. If for any $j_0 \in \mathbb{Z}$, $(c^0_{j_0}(k))_{k\in\mathbb{Z}}$ is an $ARIMA(K, D, L^0_{j_0})$, $D \in \mathbb{N}$, then, for $j > j_0$, the approximation sequence $(c^0_j(k))_{k\in\mathbb{Z}}$ is an $ARIMA(K, D, L^0_j)$, with*

$$L^0_j \le \bar{L}_j \triangleq (K + D + \frac{P}{M-1})(1 - M^{j_0-j}) + L^0_{j_0} M^{j_0-j}\,. \tag{32}$$

*If $r \le D-1$, the wavelet sequences $(c^p_j(k))_{k\in\mathbb{Z}}$ are $ARIMA(K, D-r, L^p_j)$ with*

$$L^p_j \le \bar{L}_j - r\,, \tag{33}$$

*and, if $r \ge D$, they reduce to $ARMA(K, L^p_j)$ processes, with*

$$L^p_j \le \bar{L}_j - D\,. \tag{34}$$

It is worth noting that the (intrascale) stationarity of the wavelet coefficients holds when the vanishing order of the analyzing wavelets is sufficiently high. We also deduce from the above statement that the order of the AR part of an ARIMA process is preserved in the decomposition, whereas the order of the MA part is dependent on $j$.

## 7. References


[1] A. Cohen, J. Froment, and J. Istas, "Analyse multirésolution des signaux aléatoires," *C. R. Acad. Sci. Paris*, vol. 312, Série I, pp. 567–570, 1991.

[2] M. Basseville, M. Benveniste, A. Chou, K. Golden, R. Nikoukhah, and A. S. Willsky, "Modeling and estimation of multiresolution stochastic processes," *IEEE Trans. Informat. Theory*, vol. IT-38, pp. 529–532, Mar. 1992.

[3] V. P. Sathe and P. P. Vaidyanathan, "Effects of multirate systems on the statistical properties of random signals," *IEEE Trans. Signal Processing*, vol. 41, pp. 131–146, Jan. 1993.

[4] R. W. Dijkerman and R. R. Mazumdar, "Wavelet representations of stochastic processes and multiresolution stochastic models," *IEEE Trans. Signal Processing*, vol. 42, pp. 1640–1652, Jul. 1994.

[5] S. Cambanis and E. Masry, "Wavelet approximations of deterministic and random signals: convergence properties and rate," *IEEE Trans. Informat. Theory*, vol. IT-40, pp. 1003–1012, Jul. 1994.

[6] H. Krim and J.-C. Pesquet, "Multiresolution analysis of a class of nonstationary processes." to appear in IEEE Trans. Informat. Theory, 1994.

[7] P. Steffen, P. N. Heller, R. A. Gopinath, and G. S. Burrus, "Theory of regular $M$-band wavelet bases," *IEEE Trans. Signal Processing*, vol. SP-41, pp. 3497–3511, Dec. 1993.

[8] I. Daubechies, *Ten Lectures on Wavelets.* Philadelphia, USA: CBMS-NSF, SIAM, 1992.


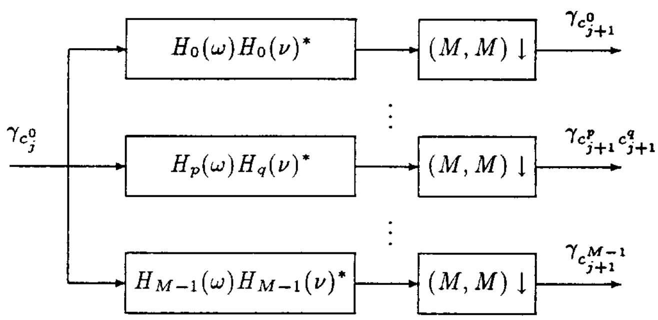


Figure 1: Recursive computation of intrascale correlations, in the nonstationary case.

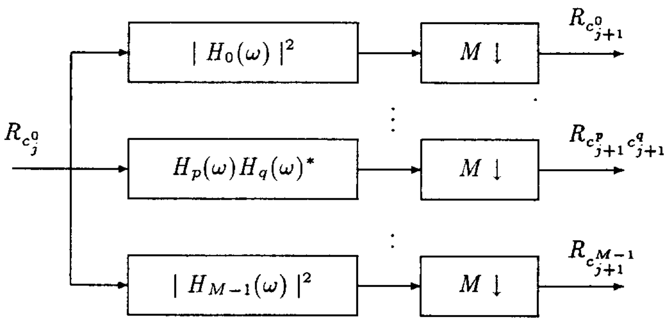


Figure 2: Recursive computation of intrascale correlations, in the stationary case.