ORTHONORMAL BASES OF BANDLIMITED WAVELETS WITH FREQUENCY OVERLAP

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ABSTRACT

Motivated by their utility in coding of digital communication signals, numerous examples of orthonormal wavelet sets that are generated from a bandlimited mother wavelet have been constructed. In most of these examples, orthogonality of the replicates at different scales is achieved by ensuring that the frequency domain supports at every scale are disjoint. Given the mother wavelet for such a set, this paper describes how it can be extended to yield a new bandlimited mother wavelet which generates an orthonormal set and whose dilated replicates overlap in the frequency domain. The utility of these wavelets to improve robustness of wavelet-coded communication signals with respect to demodulation by an unintended receiver is also discussed. The possibility of developing a new class of symbols by unitary warping of the frequency axis is also considered, but is shown not to be viable.

1. INTRODUCTION

Wavelet modulation techniques for encoding communication data into signals that are robust with respect to interference in cluttered communication environments, intentional jamming, and interception by unintended receivers have been introduced within the past three years [1, 2, 3, 4, 5]. In particular, the idea of multiple-access techniques in which digital communication signals are encoded on orthogonal wavelet sets was introduced in [5]. The desirability of orthonormal wavelet sets generated by bandlimited mother wavelets for use as "symbols" in such a scheme is discussed in [6].

Techniques for constructing bandlimited mother wavelets that generate orthonormal sets are also developed in [6]. To achieve orthogonality of the replicates at different scales of dilation, these methods ensure that replicates at two different scales do not overlap in the frequency domain. The purpose of this paper is to present a method for constructing bandlimited mother wavelets which (i) generate orthonormal sets and (ii) whose replicates at different scales overlap in the frequency domain. The method presented here draws on a technique described in [7]. The orthonormal wavelet set generated by the "overlapping" mother wavelet is shown to be a basis if and only if the set generated by the original mother wavelet is a basis.

The overlapped bandlimited orthonormal wavelet sets constructed have certain robustness features if used as symbols in multiple-access broadcast communication systems. These are also discussed.

A technique to produce a new orthonormal basis by applying a unitary warping transformation to the elements of another orthonormal basis was presented in [9, 10]. The possibility of using this technique to develop new classes of wavelet symbols by unitary warping of the frequency axis is considered, but is shown not to be viable because frequency warping does not commute with the operations of time shifting and dilation.

2. SCALE DIVISION MULTIPLE ACCESS

A multiple-access communication scheme based on the theory of wavelets introduced in [5] encodes bits on orthogonal symbols obtained by time shifting and dilating a single mother wavelet symbol, which may be chosen from a fairly broad class of wavelets. Separation of the various users' messages into orthogonal channels at different dilations or "scales" suggests the name "scale division multiple access" (SDMA). The spectral density of an SDMA signal depends on the mother wavelet and the statistical structure of the message sequence. Choosing wavelets with broad or complicated spectra will spread the spectrum of the SDMA signal, thereby

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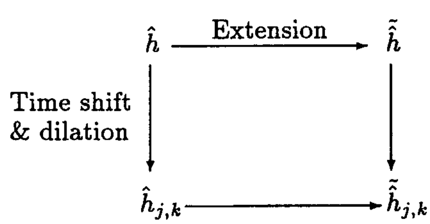
contributing robustness with respect to detection, interference, and narrowband jamming.

Constructions of bandlimited orthonormal wavelets are presented in [8] and the use of such wavelets for use as symbols in SDMA is discussed in [5]. Despite having complicated band structures, the wavelet symbols described in [5, 6] rely on the dilated replicates having nonoverlapping frequency supports for orthogonality. Thus, the channels can be separated by an unintended receiver using a relatively simple receiver structure (i.e., a bank of bandpass filters). The following section of this summary sketches a method for introducing frequency overlap into previously nonoverlapping wavelet symbols while retaining orthonormality.

3. CONSTRUCTION OF OVERLAPPED BANDLIMITED ORTHONORMAL WAVELET SETS

The construction developed in this section draws on a method presented in [7]. This method assumes the real line has been partitioned into disjoint intervals I_j, $j \in \mathbb{Z}$. If $\{f_{j,k} | k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(I_j)$, then $\{f_{j,k} | j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. A new orthonormal basis is constructed by extension of the functions $f_{j,k}$ off the interval I_j in a specific way, resulting in a basis in which the elements "overlap."

The goal of this section is to indicate how the technique introduced in [7] can be applied to the Fourier transform of a single mother wavelet in such a way that its support is expanded and orthonormality of its dyadically dilated and time-shifted replicates is retained. Specifically, the diagram



(in which $\hat{h}$ denotes the Fourier transform of the wavelet h) should be commutative. This goal is addressed as follows. Let h be a bandlimited mother wavelet having the properties:

1. Its Fourier transform $\hat{h}$ is real-valued and even. The steps in the construction described below apply to the portion of $\hat{h}$ on the positive frequency axis and are assumed to also be applied to the portion of $\hat{h}$ on the negative frequency axis to maintain even symmetry.
2. The support of $\hat{h}$ is "orthogonal" in the sense of [6]; i.e., the portion on the positive frequency axis is of the form

$$[2^{d_1}\pi, 2^{d_1}a_1] \cup \cdots \cup [2^{d_n}a_{n-1}, 2^{d_n}2\pi]$$

where $n \geq 1$, $\pi = a_0 < a_1 < \cdots < a_{n-1} < a_n = 2a_0 = 2\pi$, and $d_1, ..., d_n$ are distinct integers.

For integers j and k, define

$$h_{j,k}(t) = 2^{-\frac{j}{2}} h\left(\frac{t - 2^j k}{2^j}\right)$$

and denote by $\hat{h}_{j,k}$ the Fourier transform of $h_{j,k}$.

Suppose $\{h_{j,k} | j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ (a class of such bases is constructed in [6]). Let

$$\begin{aligned} \epsilon_0 &< \min\{(a_1 - a_0)/2, (a_n - a_{n-1})/2\} \\ \epsilon_i &< \min\{(a_i - a_{i-1})/2, (a_{i+1} - a_i)/2\} \\ & \quad i = 1, \ldots, n-1 \\ \epsilon_n &= 2\epsilon_0 \end{aligned}$$

Define an extension $\tilde{\hat{h}}$ of $\hat{h}$ by

1. Constructing the odd extension of $\hat{h}$ about the point $2^{d_i}a_{i-1}$ into the interval

$$[2^{d_i}(a_{i-1} - \epsilon_{i-1}), 2^{d_i}a_{i-1}] \quad i = 1, \ldots, n;$$

2. Constructing the even extension of $\hat{h}$ about the point $2^{d_i}a_i$ into the interval

$$[2^{d_i}a_{i-1}, 2^{d_i}(a_i + \epsilon_i)] \quad i = 1, \ldots, n;$$

and

3. Repeating these steps on the negative frequency axis to preserve even symmetry.

Denote the extended wavelet just constructed by $\tilde{\hat{h}}$ and let $\hat{w}$ denote a frequency domain window function with support identical to that of $\tilde{\hat{h}}$ and with the amplitude normalization properties described in [7]. An example of $\tilde{\hat{h}}(\omega)$ using

$$\hat{h}(\omega) = \begin{cases} 1 & \pi \leq |\omega| < 2\pi \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

and an example of $\hat{\omega}$ is depicted in figure 1.

Theorem: $\hat{u}(\omega) = \hat{w}(\omega)\tilde{\hat{h}}(\omega)$ is a bandlimited orthonormal wavelet basis of $L^2(\mathbb{R})$.

Proof: Since, in general, the mother wavelet used here, $\hat{h}(\omega)$, occupies n disjoint bands on the positive frequency axis, the same arguments of the proof in [7]

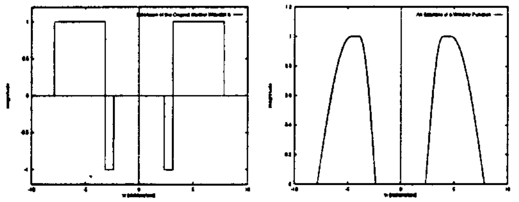


Figure 1: An extended mother wavelet, $\tilde{\hat{h}}(\omega)$, using the above $\hat{h}(\omega)$ and an example of normalized frequency domain window function, $\hat{w}(\omega)$.

can be applied to each band. Also, those steps can be applied to the bands on the negative frequency axis preserving even symmetry and the proof is complete.

Note that, by construction, the dilated replicates of $\hat{u}$ have non-trivial overlap. Figure 2 shows $\hat{u}(\omega)$, a particular overlapped bandlimited orthonormal mother wavelet constructed using the above example.

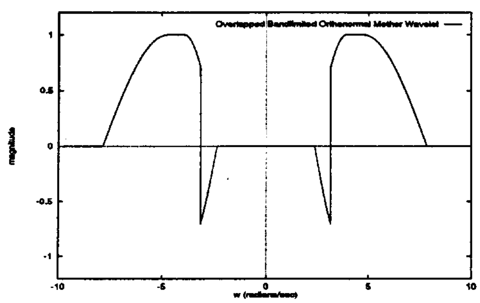


Figure 2: An example of overlapped bandlimited orthonormal mother wavelet, $\hat{u}(\omega)$.

4. UNITARY TRANSFORMATION

A technique to produce a new orthonormal basis by applying a unitary warping transformation to the elements another orthonormal basis was presented in [9, 10]. The possibility of constructing a new bandlimited mother wavelet symbol using this warping technique in the frequency domain is shown to be impossible by the following result.

Theorem: Let $\hat{h}$ be a bandlimited orthonormal mother wavelet and consider a unitary frequency axis warping U by a differentiable function γ; i.e.,

$$(U\hat{h})(\omega) = |\gamma'(\omega)|^{1/2}\hat{h}[\gamma(\omega)]$$

Only the identity warping defined by $\gamma(\omega) = \omega$ yields an orthonormal wavelet basis.

Proof: Applying U to each basis function $\hat{h}_{j,k}$ yields a new orthonormal family, the elements of which are defined by

$$(U\hat{h}_{j,k})(\omega) = \sqrt{|\gamma'(\omega)|}2^{\frac{j}{2}}e^{-i2^j k\gamma(\omega)}\hat{h}[2^j\gamma(\omega)] \quad (2)$$

Applying U to a single wavelet yields a family of dilated and translated replicates defined by

$$(U\hat{h})_{j,k}(\omega) = \sqrt{|\gamma'(2^j\omega)|}2^{\frac{j}{2}}e^{-i2^j k\omega}\hat{h}[\gamma(2^j\omega)] \quad (3)$$

Comparing equations (2) and (3), it is evident that the only warping function for which they will agree is the identity warping $\gamma(\omega) = \omega$.

5. CONCLUSION

A technique for introducing frequency overlap into a fairly rich class of known bandlimited orthonormal wavelet sets has been described. If used as symbols in a wavelet-based multiple access communication system, these wavelets should improve robustness with respect to demodulation by an unintended receiver. The possibility of constructing new wavelet symbols using unitary frequency warping has also been investigated.

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