ON THE EQUIVALENCE OF GENERALIZED JOINT SIGNAL REPRESENTATIONS

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ABSTRACT

Joint time-frequency representations have proven very useful for analyzing signals in terms of time-frequency content. Recently, in an attempt to tailor joint representations to a richer class of signals, two approaches (Cohen's and Baraniuk's) to obtaining joint representations of arbitrary variables have been proposed. Baraniuk's generalization appears broader than Cohen's, since the latter can be recovered from the former as a special case. One of the main results of this paper is that, despite being apparently quite different, the two approaches to generalized joint representations are *exactly* equivalent. We explicitly characterize the mapping which relates the representations of the two methods, and also determine the corresponding relationship between the operators of the two methods. A practical implication of the results is that one can avoid the group transforms in Baraniuk's approach, which may not be computationally efficient, by replacing them with Fourier transforms in Cohen's method.

1. INTRODUCTION

Joint time-frequency representations (TFRs) are used for analyzing signals in terms of time and frequency content, or signals with unknown time or frequency shifts. They have proven very useful for many types of signals; radar, speech and transients are some examples. Recently, however, joint representations in terms of other variables have been extensively studied in an attempt to tailor the representations to a richer class of signals [1, 2, 3, 4, 5, 6]. The wavelet transform and generalizations are the best known, which analyze signals in terms of time and scale content.

Recently, Cohen and Baraniuk proposed methods for obtaining joint representations of arbitrary variables (signal parameters) [2, 3, 5]. The ability to tailor new representations to any "natural" signal parameters or characteristics of interest is potentially of great value in many applications. The fundamental idea behind both approaches is associating variables of interest with appropriate operators. Cohen's method is based on Hermitian operator correspondence, that is, associating variables with *Hermitian* (self-adjoint) operators, while Baraniuk's method is based on associating variables with parameterized *unitary* operators which are unitary representations of certain 1d groups [5]. On the surface, Baraniuk's method appears to be more general than Cohen's approach, since Cohen's method can be recovered from it by basing the construction on the translational group of reals. Cohen's approach, on the other hand, generally appears more attractive computationally by virtue of being based on Fourier transforms as opposed to arbitrary group transforms as in Baraniuk's approach. Clarification of the relationship between these two methods is essential for a better understanding of generalized joint signal representations.

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We show in this paper that, despite the apparent differences between Cohen's and Baraniuk's approaches, the two methods are *exactly equivalent*. We prove that there is a one-to-one and onto mapping which relates the joint representations constructed by the two methods. In addition to explicitly characterizing this mapping, we also derive equations which explicitly state the relationship between the unitary operators of Baraniuk's method and the Hermitian operators of Cohen's approach.

In addition to their theoretical significance, the results have important practical implications as well. One particular implication is that we can do away with the group transforms in Baraniuk's approach, which may not be computationally efficient, by replacing them with Fourier transforms in Cohen's method. However, Baraniuk's method often offers a more obvious means of deriving generalized signal representations based on desired signal parameters. Unification of these concepts should enhance both conceptual understanding and practical application of these methods.

After briefly describing the two methods for generating joint distributions of arbitrary variables in the next section, we present the main results in section 3. We summarize the results and comment on their implications in section 4.

2. PRELIMINARIES

To be able to present the results of the paper, we need a description for the two methods for generalized joint signal representations. For simplicity, we will consider joint distributions of two variables only; extension to multiple variables is straightforward. In the following we use the subscript "H" for Hermitian operators, operators corresponding to Baraniuk's approach wear a hat "$\wedge$", and the *dual* operator of an operator is distinguished by the superscript "$\diamond$."

2.1. Cohen's approach: Hermitian operator correspondence

Let a and b be two variables of interest; they could be time and scale, for example. In Cohen's approach, the variables, a and b, are associated with appropriate Hermitian operators, $\mathcal{A}_H$ and $\mathcal{B}_H$, respectively. The eigenfunctions of $\mathcal{A}_H$ and $\mathcal{B}_H$ define unitary signal representations S_A and S_B which

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yield the a- and b-representation of the signal. The joint distribution/representation, $(Ps)(a,b)$, should satisfy the a and b marginals; that is,

$$\int (Ps)(a,b)db = |(S_{\mathcal{A}}s)(a)|^2 , \quad (1)$$

$$\int (Ps)(a,b)da = |(S_{\mathcal{B}}s)(b)|^2 \quad (2)$$

The idea behind generating such distributions is that they can be recovered from the characteristic function Ms:

$$(Ps)(a,b) = \int\int (Ms)(\theta,\tau)e^{-j2\pi\theta a}e^{-j2\pi\tau b}d\theta d\tau, \quad (3)$$

$$(Ms)(\theta,\tau) = \int\int (Ps)(a,b)e^{j2\pi\theta a}e^{j2\pi\tau b}dadb . \quad (4)$$

Note that Ps and Ms are related through a 2d Fourier transform. The key observation is that the characteristic function, being an *average* of $e^{j2\pi\theta a}e^{j2\pi\tau b}$, can be directly computed from the signal using a characteristic function operator, $\mathcal{M}_{(\theta,\tau)}$, an example of which is $\mathcal{M}_{(\theta,\tau)} = e^{j2\pi\theta\mathcal{A}_H}e^{j2\pi\tau\mathcal{B}_H}$ [7, 2] (there are infinitely many characteristic function operators, in general.). Specifically, Ms can be computed as

$$(Ms)(\theta,\tau) = \langle \mathcal{M}_{(\theta,\tau)}s, s\rangle = \int (\mathcal{M}_{(\theta,\tau)}s)(t)s^*(t)dt \quad (5)$$

$$= \phi(\theta,\tau)\langle e^{j2\pi\theta\mathcal{A}_H}e^{j2\pi\tau\mathcal{B}_H}s, s\rangle , \quad (6)$$

where ϕ is some 2d kernel, and then, $P(\phi)s$ can be recovered via (3).

2.2. Baraniuk's approach: unitary operator correspondence

Baraniuk's approach is based on associating variables with parameterized operators which are unitary representations of some 1d locally compact abelian (LCA) group. Specifically, let G be a 1d LCA group with group operation $\bullet$.[1] We will use the symbols a, b for elements of G. A complex-valued function α on G is called a *character* of G if $|\alpha(a)| = 1$ $\forall a \in G$, and if it satisfies the functional equation $\alpha(a\bullet b) = \alpha(a)\alpha(b)$ $\forall a,b \in G$ [8]. The set of all continuous characters of G itself forms a 1d LCA group Γ, the *dual* group of G, with the group operation $\circ$ defined by $(\alpha\circ\beta)(a) = \alpha(a)\beta(a)$, $a \in G$, $\alpha,\beta \in \Gamma$. Because of this duality, it is convenient to use the following notation for characters: $\alpha(a) \equiv (a,\alpha) \equiv (a,\alpha)_G$.

The natural signal space associated with the group G is $\mathcal{H}_1 = L^2(G, d\mu_G)$ where μ_G is the Haar measure associated with G, and the natural analogue of the Fourier transform is the unitary group Fourier transform, $\mathbb{F}_G : \mathcal{H}_1 \to L^2(\Gamma, d\mu_\Gamma) = \mathcal{H}_2$, based on the characters and defined as

$$(\mathbb{F}_G s)(\alpha) = \int_G s(a)(a,\alpha)^* d\mu_G(a) , \quad (7)$$

$$(\mathbb{F}_G{}^{-1}h)(a) = \int_\Gamma h(\alpha)(a,\alpha)d\mu_\Gamma(\alpha) . \quad (8)$$

[1] An abelian group is a set G in which a binary operation $\bullet$ is defined, with the following properties: 1) $x\bullet y = y\bullet x$, $\forall x,y \in G$, 2) $x\bullet(y\bullet z) = (x\bullet y)\bullet z$, $\forall x,y,z \in G$, 3) there exists an identity element $0 \in G$ such that $x\bullet 0 = x$ $\forall x \in G$, and 4) for each $x \in G$ there exists an inverse $x^{-1} \in G$ such that $x\bullet x^{-1} = 0$ [8].

A parameterized unitary operator $\widehat{\mathcal{A}}_a$, $a \in G$, is a unitary representation of G on $\mathcal{H}_1$ if it satisfies $\widehat{\mathcal{A}}_a\widehat{\mathcal{A}}_b = \widehat{\mathcal{A}}_{a\bullet b}$. With each such $\widehat{\mathcal{A}}_a$ we can associate two unitary signal representations. The first one, $S_{\widehat{\mathcal{A}}} : \mathcal{H}_1 \to \mathcal{H}_2$, based on the eigenfunction of $\widehat{\mathcal{A}}_a$ is $\widehat{\mathcal{A}}$-invariant [5, 6]; that is, $|(S_{\widehat{\mathcal{A}}}\widehat{\mathcal{A}}_a s)(\alpha)| = |(S_{\widehat{\mathcal{A}}}s)(\alpha)|$. The other one, $S_{\widehat{\mathcal{A}}^\circ} : \mathcal{H}_1 \to \mathcal{H}_1$, based on the eigenfunctions of the *dual* operator of $\widehat{\mathcal{A}}_a$ (defined in section 3.1) [6], is $\widehat{\mathcal{A}}$-covariant; that is, $(S_{\widehat{\mathcal{A}}^\circ}\widehat{\mathcal{A}}_a s)(b) = (S_{\widehat{\mathcal{A}}^\circ}s)(b\bullet a)$. $|(S_{\widehat{\mathcal{A}}}s)(\alpha)|^2$ and $|(S_{\widehat{\mathcal{A}}^\circ}s)(b)|^2$ are referred to as "invariant energy density (IED)" and "covariant energy density (CED)", respectively, in [5].

Now suppose we are interested in the joint distributions of variables associated with two unitary operators $\widehat{\mathcal{A}}_a$ and $\widehat{\mathcal{B}}_b$ (the nature of this correspondence with variables is not clear in [5]; using the concepts of "duality" and "shift operators", we make this correspondence precise in [9].). Baraniuk's approach allows us to recover either the IED or CED marginal corresponding to an operator; that is, a joint distribution $\widehat{P}s$ satisfies

$$\int (\widehat{P}s)(u,v)d\mu_{\widehat{\mathcal{B}}}(v) = |\overline{S}_{\widehat{\mathcal{A}}}(u)|^2 , \quad (9)$$

$$\int (\widehat{P}s)(u,v)d\mu_{\widehat{\mathcal{A}}}(u) = |\overline{S}_{\widehat{\mathcal{B}}}(v)|^2 , \quad (10)$$

where the measures are either μ_G or μ_Γ, and $\overline{S}_{\widehat{\mathcal{A}}}$ is either $S_{\widehat{\mathcal{A}}}$ (IED) or $S_{\widehat{\mathcal{A}}^\circ}$ (CED) ($\overline{S}_{\widehat{\mathcal{B}}}$ is defined similarly) [5]. In this approach too, a modification of the characteristic function method is used. If the $\widehat{\mathcal{A}}$-IED marginal is desired, define the unitary operator $\overline{\mathcal{A}}_a = \widehat{\mathcal{A}}_a$, and if the CED marginal is desired define $\overline{\mathcal{A}}_\alpha = \widehat{\mathcal{A}}^\circ_\alpha = S^{-1}_{\widehat{\mathcal{A}}^\circ}\widehat{\Lambda}^\circ_\alpha S_{\widehat{\mathcal{A}}^\circ}$, the dual operator of $\widehat{\mathcal{A}}_a$, where $(\widehat{\Lambda}^\circ_\alpha s)(a) = (a,\alpha)^* s(a)$. Similarly define $\overline{\mathcal{B}}_b$ and $\overline{\mathcal{B}}_\beta$. Since there are four different combinations, corresponding to different marginals, we illustrate with a specific case parallel to [5]. Suppose we are interested in $\widehat{\mathcal{A}}$-CED and $\widehat{\mathcal{B}}$-IED marginals. Then, the characteristic function is computed as

$$(\widehat{M}s)(\alpha,b) = \widehat{\phi}(\alpha,b)\langle\overline{\mathcal{A}}_\alpha\overline{\mathcal{B}}_b s, s\rangle \quad (11)$$

and the distribution can be recovered (using $\mathbb{F}_G$) as

$$(\widehat{P}(\widehat{\phi})s)(a,\beta) = \int_G\int_\Gamma (\widehat{M}s)(\alpha,b)(a,\alpha)(b,\beta)^* d\mu_\Gamma(\alpha)d\mu_G(b) , \quad (12)$$

which yields the $\widehat{\mathcal{A}}$-CED and $\widehat{\mathcal{B}}$-IED marginals [5].

3. MAIN RESULTS

In the last section we described Cohen's and Baraniuk's approaches to generalized joint signal representations. Baraniuk's generalization, based on unitary representations of arbitrary 1d LCA groups, is apparently broader than Cohen's. In this section we present the main results of the paper which show that, despite the apparent differences between them, the two approaches are exactly equivalent. The exact relationship between the operators of the two methods is also characterized. But we first we need to define the notion of dual operators which is done next. Throughout this section, the basic setup is the same as in section 2.2.

3.1. Dual operators

Given the Hilbert space $\mathcal{H}_1$, we can naturally define two types of parameterized unitary operators: the representations of G on $\mathcal{H}_1$, and the representations of the dual group, Γ, on $\mathcal{H}_1$. In Baraniuk's method, the latter type of operators are used for computing the characteristic function if a CED marginal is desired. Let $\widehat{\mathcal{A}}_a$, $a \in G$, and $\widehat{\mathcal{B}}^\diamond_\beta$, $\beta \in \Gamma$, be unitary representations of G and Γ, respectively, on $\mathcal{H}_1$; that is, $\widehat{\mathcal{A}}_a\widehat{\mathcal{A}}_{a'} = \widehat{\mathcal{A}}_{a \bullet a'}$ and $\widehat{\mathcal{B}}^\diamond_\beta\widehat{\mathcal{B}}^\diamond_{\beta'} = \widehat{\mathcal{B}}^\diamond_{\beta \circ \beta'}$. $\widehat{\mathcal{A}}_a$ and $\widehat{\mathcal{B}}^\diamond_\beta$ admit the following formal spectral representations [10, 9]: $\widehat{\mathcal{A}}_a = S^{-1}_{\widehat{\mathcal{A}}}\widehat{\Lambda}_a S_{\widehat{\mathcal{A}}}$ and $\widehat{\mathcal{B}}^\diamond_\beta = S^{-1}_{\widehat{\mathcal{B}}^\diamond}\widehat{\Lambda}^\diamond_\beta S_{\widehat{\mathcal{B}}^\diamond}$ where $S_{\widehat{\mathcal{A}}} : \mathcal{H}_1 \to \mathcal{H}_2$ and $S_{\widehat{\mathcal{B}}^\diamond} : \mathcal{H}_1 \to \mathcal{H}_1$ are isometries based on the eigenfunctions of $\widehat{\mathcal{A}}_a$ and $\widehat{\mathcal{B}}^\diamond_\beta$, and $(\widehat{\Lambda}_a s)(\gamma) = (a,\gamma)s(\gamma)$, $s \in \mathcal{H}_2$, and $(\widehat{\Lambda}^\diamond_\beta s)(b) = (b,\beta)^* s(b)$, $s \in \mathcal{H}_1$. Also, by Stone's theorem [10], there exist unique Hermitian operators, $\widehat{\mathcal{A}}_H$ and $\widehat{\mathcal{B}}_H$, defined on $\mathcal{H}_1$, such that formally $\widehat{\mathcal{A}}_a = (a, \widehat{\mathcal{A}}_H)$ and $\widehat{\mathcal{B}}^\diamond_\beta = (\widehat{\mathcal{B}}^\diamond_H, \beta)^*$. Dual operators are a special pair of such operators.

Defintion: Dual operators. *The operators $\widehat{\mathcal{A}}_a$ ($\widehat{\mathcal{A}}_H$) and $\widehat{\mathcal{B}}^\diamond_\beta$ ($\widehat{\mathcal{B}}^\diamond_H$) are* **dual** *if $S_{\widehat{\mathcal{A}}} = \mathbb{F}_G S_{\widehat{\mathcal{B}}^\diamond}$. In such a case, we denote $\widehat{\mathcal{B}}^\diamond_\beta$ by $\widehat{\mathcal{A}}^\diamond_\beta$ ($\widehat{\mathcal{B}}^\diamond_H$ by $\widehat{\mathcal{A}}^\diamond_H$).*

3.2. Equivalence results

As we mentioned in the introduction, Baraniuk's approach is implicitly based on a class of 1d LCA groups which are isomorphic to the translation group of reals, $\mathbb{R}$, with the group operation being addition (the dual group of $(\mathbb{R},+)$ is $(\mathbb{R},+)$). Let G be the underlying group in Baraniuk's construction. Then, by assumption, there exists an *isomorphism*[2] $\psi : G \to \mathbb{R}$ (onto $\mathbb{R}$) such that $\psi(a \bullet b) = \psi(a) + \psi(b)$ [8]. It follows that the dual groups must also be isomorphic; that is, there exists an isomorphism $\varphi : \Gamma \to \mathbb{R}$ (onto $\mathbb{R}$) such that $\varphi(\alpha \circ \beta) = \varphi(\alpha) + \varphi(\beta)$. Although the dual isomorphism is not unique, after imposing certain normalizations on the measures μ_G and μ_Γ, $d\mu_G(\psi^{-1}(x)) = dx$ in particular, the following theorem is proved in [9] which characterizes a particular one.

Theorem 1. *For each $\alpha \in \Gamma$, define $\varphi(\alpha) \in \mathbb{R}$ as*

$$e^{+j2\pi b\varphi(\alpha)} \equiv (b, \varphi(\alpha))_{\mathbb{R}} = (\psi^{-1}(b), \alpha)_G \ , \quad b \in \mathbb{R} \ . \quad (13)$$

Then, the functional equation (13) defines an isomorphism $\varphi : \Gamma \to \mathbb{R}$ which is onto $\mathbb{R}$ and satisfies

$$d\mu_\Gamma(\varphi^{-1}(y)) = dy \quad \text{for all } y \in \mathbb{R} \ . \quad (14)$$

Note that the functional equation (13) relates the characters of $(G, \bullet)$ to those of $(\mathbb{R},+)$, the complex exponentials. The isomorphisms, ψ and φ, are central to all the main results.

Now let $\widehat{\mathcal{A}}_a$ and $\widehat{\mathcal{B}}_b$ be two unitary operators in Baraniuk's approach corresponding to the variables whose joint representations are desired. Recall that $\widehat{\mathcal{A}}_a$ and $\widehat{\mathcal{B}}_b$ are unitary representations of G on $\mathcal{H}_1$. Define the mapping $T_\psi : L^2(\mathbb{R}, dx) \to L^2(G, d\mu_G)$ as

$$(T_\psi s)(a) = s(\psi(a)) \ , \quad a \in G \ . \quad (15)$$

[2] An isomorphism is a one-to-one mapping between two sets.

It is easy to verify that T_ψ is an isometry from $L^2(\mathbb{R}, dx)$ onto $L^2(G, d\mu_G)$ [9]; that is, $\|T_\psi s\|_{\mathcal{H}_1} = \|s\|_{L^2}$. Similarly, define the isometry $T_\varphi : L^2(\mathbb{R}, dx) \to L^2(\Gamma, d\mu_\Gamma)$ as $(T_\varphi s)(\alpha) = s(\varphi(\alpha))$. The mapping T_ψ allows us to map unitary representations of G on $\mathcal{H}_1$ to those of $\mathbb{R}$ on $L^2(\mathbb{R}, dx)$. Define an operator on $L^2(\mathbb{R}, dx)$ as

$$\mathcal{A}_{\psi(a)} = T_\psi^{-1}\widehat{\mathcal{A}}_a T_\psi \ , \quad (16)$$

which is a unitary representation of $(\mathbb{R},+)$ on $L^2(\mathbb{R}, dx)$ [9]; that is, $\mathcal{A}_x\mathcal{A}_y = \mathcal{A}_{x+y}$, for all $x, y \in \mathbb{R}$. Similarly, if we have a unitary representation of Γ on $\mathcal{H}_1$, $\widehat{\mathcal{A}}^\diamond_\alpha$, then the operator

$$\mathcal{A}^\diamond_y = T_\psi^{-1}\widehat{\mathcal{A}}^\diamond_{\varphi^{-1}(y)} T_\psi \quad (17)$$

is also a representation of $\mathbb{R}$ (the dual group!) on $L^2(\mathbb{R}, dx)$. Given $\mathcal{A}_x$ and $\mathcal{A}^\diamond_y$, by Stone's theorem there exist unique Hermitian operators $\mathcal{A}_H$ and $\mathcal{A}^\diamond_H$ defined on $L^2(\mathbb{R}, dx)$ such that

$$\mathcal{A}_x = e^{j2\pi x\mathcal{A}_H} = (x, \mathcal{A}_H)_{\mathbb{R}} \quad (18)$$

$$\mathcal{A}^\diamond_y = e^{-j2\pi y\mathcal{A}^\diamond_H} = (\mathcal{A}^\diamond_H, y)^*_{\mathbb{R}} \ . \quad (19)$$

Now we are in a position to prove the main result of the paper. Since there are four types of joint representations in Baraniuk's approach, corresponding to the choice of marginals, we characterize the equivalence for representations with one CED and one IED marginal; the equivalence for the remaining types can be readily inferred from the stated result.

Theorem 2. *Let $(G, \bullet)$ be a one-parameter LCA group isomorphic to $(\mathbb{R},+)$. For each $\widehat{P}$ from Baraniuk's class of joint signal representations corresponding to operators $\widehat{\mathcal{A}}_a$ and $\widehat{\mathcal{B}}_b$ and yielding $\widehat{\mathcal{A}}$-CED and $\widehat{\mathcal{B}}$-IED marginals, there exists a P in a corresponding Cohen's class (associated with a pair of Hermitian operators) of joint signal representations (and vice versa) such that*

$$(\widehat{P}(\widehat{\phi})s)(a,\beta) = (P(\phi)T_\psi^{-1}s)(\psi(a), \varphi(\beta)) \quad \text{where} \quad (20)$$

$$(T_\psi s)(a) = s(\psi(a)) \quad (21)$$

is an isometry from $L^2(\mathbb{R}. dx)$ onto $L^2(G, d\mu_G)$ and the kernels are related as

$$\widehat{\phi}(\alpha, b) = \phi(\varphi(\alpha), \psi(b)) \ . \quad (22)$$

The equivalence (20) is unitary; that is,

$$(\widehat{P}(\widehat{\phi})s)(a,\beta) = (\mathcal{V}_G P(\phi)\mathcal{U}_G s)(a,\beta) \quad (23)$$

where $\mathcal{U}_G = T_\psi^{-1}$ and $\mathcal{V}_G$ is an isometry from $L^2(\mathbb{R}^2, dx \times dx)$ onto $L^2(G \times \Gamma, d\mu_G \times d\mu_\Gamma)$: $(\mathcal{V}_G P)(a,\beta) = P(\psi(a), \varphi(\beta))$.

Proof: *From (11) and (12) we see that $\widehat{P}(\widehat{\phi})$ is given by*

$$(\widehat{P}(\widehat{\phi})s)(a,\beta) = \int_G\int_\Gamma \widehat{\phi}(\alpha, b)\langle\widehat{\mathcal{A}}^\diamond_\alpha\widehat{\mathcal{B}}_b s, s\rangle$$
$$(a,\alpha)(b,\beta)^* d\mu_\Gamma(\alpha)d\mu_G(b) \ , \quad s \in L^2(G, d\mu_G) \quad (24)$$

where we have substituted $\overline{\mathcal{A}}_\alpha = \widehat{\mathcal{A}}^\diamond_\alpha$ and $\overline{\mathcal{B}}_b = \widehat{\mathcal{B}}_b$ in (11), since $\widehat{\mathcal{A}}$-CED and $\widehat{\mathcal{B}}$-IED marginals are desired. Using (16) and (17) we get

$$(\widehat{P}(\widehat{\phi})s)(a,\beta) = \int_G\int_\Gamma \widehat{\phi}(\alpha, b)\langle\mathcal{A}^\diamond_{\varphi(\alpha)}\mathcal{B}_{\psi(b)}T_\psi^{-1}s, T_\psi^{-1}s\rangle$$
$$(a,\alpha)(b,\beta)^* d\mu_\Gamma(\alpha)d\mu_G(b) \quad (25)$$

for some unitary operators, $\mathcal{A}_x^\circ$ and $\mathcal{B}_x$, $x \in \mathbb{R}$, which are unitary representations of $(\mathbb{R},+)$ on $L^2(\mathbb{R},dx)$. Making the substitutions $b=\psi^{-1}(y)$ and $\alpha=\varphi^{-1}(x)$ in (25) yields

$$(\widehat{P}(\widehat{\phi})s)(a,\beta) = \int_{\mathbb{R}^2} \widehat{\phi}(\varphi^{-1}(x),\psi^{-1}(y))\langle \mathcal{A}_x^\circ \mathcal{B}_y T_\psi^{-1} s, T_\psi^{-1} s\rangle$$

$$(a,\varphi^{-1}(x))(\psi^{-1}(y),\beta)^* d\mu_\Gamma(\varphi^{-1}(x)) d\mu_G(\psi^{-1}(y))$$

$$= \int_{\mathbb{R}^2} \phi(x,y)\langle e^{-j2\pi x \mathcal{A}_H^\circ} e^{j2\pi y \mathcal{B}_H} T_\psi^{-1} s, T_\psi^{-1} s\rangle$$

$$(a,\varphi^{-1}(x))(\psi^{-1}(y),\beta)^* dxdy \qquad (26)$$

where in the last equality ϕ is defined as in (22), and we used (18) and (19). Finally, using the functional equation (13) in theorem 1, (26) becomes

$$(\widehat{P}(\widehat{\phi})s)(a,\beta) = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(x,y)\langle e^{-j2\pi x \mathcal{A}_H^\circ} e^{j2\pi y \mathcal{B}_H} T_\psi^{-1} s, T_\psi^{-1} s\rangle$$

$$e^{j2\pi\psi(a)x} e^{-j2\pi y \varphi(\beta)} dxdy \ . \qquad (27)$$

Comparing (27) with (6) and (3) we get the relation (20). The fact that T_ψ and $\mathcal{V}_G$ are isometries follows directly from the definitions of the isomorphisms, ψ and φ, and the normalizations imposed on them.

It is worth noting that the equivalence between the two approaches is based on *axis* transformations of the signal and the joint representation, and the axis transformations are simply the group isomorphisms, ψ and φ. Thus, we only need to figure out the group isomorphisms to go from Baraniuk's approach to Cohen's method and vice versa. Moreover, the proof of theorem 2 also relates, albeit somewhat implicitly, the operators of the two methods. The next theorem, proved in [9], explicitly states this relationship.

Theorem 3. *Let $(G,\bullet)$ be a one-parameter LCA group isomorphic to $(\mathbb{R},+)$. Let $\widehat{\mathcal{A}}_a$ and $\widehat{\mathcal{B}}_b$ be the two unitary operators in Baraniuk's approach whose joint representations, with $\widehat{\mathcal{A}}$-CED and $\widehat{\mathcal{B}}$-IED marginals, are desired. Let $\mathcal{A}_H$ and $\mathcal{B}_H$ be the corresponding Hermitian operators in the equivalent Cohen's class of joint signal representations. Then, $\mathcal{A}_H$ and $\widehat{\mathcal{A}}_a$ are related by*

$$\mathcal{A}_H = S_{\mathcal{A}}^{-1} \Lambda S_{\mathcal{A}} \qquad (28)$$

where Λ is defined on $L^2(\mathbb{R},dx)$ by $(\Lambda s)(x) = xs(x)$ and the eigenfunctions of $\mathcal{A}_H$ are related to those of $\widehat{\mathcal{A}}_a = S_{\widehat{\mathcal{A}}}^{-1}\widehat{\Lambda}_a S_{\widehat{\mathcal{A}}}$ by

$$S_{\mathcal{A}} = T_\psi^{-1} S_{\widehat{\mathcal{A}}^\circ} T_\psi \ , \qquad (29)$$

where $S_{\widehat{\mathcal{A}}^\circ} = \mathbb{F}_G{}^{-1} S_{\widehat{\mathcal{A}}}$ are the eigenfunctions of $\overline{\mathcal{A}}_\alpha = \widehat{\mathcal{A}}_\alpha^\circ = S_{\widehat{\mathcal{A}}^\circ}^{-1}\widehat{\Lambda}_\alpha^\circ S_{\widehat{\mathcal{A}}^\circ}$, the dual operator of $\widehat{\mathcal{A}}_a$. Similarly, the operators $\mathcal{B}_H$ and $\widehat{\mathcal{B}}_b$, corresponding to the other variable, are related by

$$\mathcal{B}_H = S_{\mathcal{B}}^{-1} \Lambda S_{\mathcal{B}} \ , \qquad (30)$$

and the eigenfunctions of $\mathcal{B}_H$ are related to those of $\widehat{\mathcal{B}}_b = S_{\widehat{\mathcal{B}}}^{-1}\widehat{\Lambda}_b S_{\widehat{\mathcal{B}}}$ by

$$S_{\mathcal{B}} = T_\varphi^{-1} S_{\widehat{\mathcal{B}}} T_\psi \ . \qquad (31)$$

Example. Let $(G,\bullet) = (\mathbb{R}_+,*)$, where $*$ denotes multiplication. One characterization of the dual group is $(\mathbb{R},+)$ with the characters given by $(a,\alpha) = e^{j2\pi\alpha\ln(a)}$. In this case, it can be easily verified that $\psi(a) = \ln(a)$, $\varphi(\alpha)=\alpha$, and the joint representations are related as characterized in theorems 1 and 2.

4. CONCLUSIONS

From a theoretical viewpoint, the results presented in the paper are significant in that they unify two apparently different approaches to joint distributions of arbitrary variables by demonstrating their equivalence and explicitly deriving the relationship between them. The results imply that by simply axis warping a given joint representation we can obtain a new representation with radically different properties.

The results are also important from a practical perspective because the equivalence shows that we can avoid computing the group transforms in Baraniuk's approach, which may not be computationally efficient, by replacing them with Fourier transforms in Cohen's method. However, we emphasize that the unitary operator method may conceptually be the preferred one in many situations. Both approaches have their merits, and by characterizing the exact relationship between them the results in this paper allow adoption of the most convenient approach in any given situation.

5. REFERENCES

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