Hermite-like reduction method for linear phase perfect reconstruction filter bank design *

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Abstract

Complete parametrization of multiband linear phase biorthogonal filter banks are given. The method uses matrix reduction methods similar to the Hermite reduction method of linear system theory. Computational algorithms are derived for design, and examples are worked out.

1 Introduction

Complete parametrization of special classes of perfect reconstruction (PR) filter banks, e.g., paraunitary and/or linear phase filter banks is an important problem because of their interests in designing optimal filters belonging that class. In spite of considerable amount of research in recent years, the problem of complete parametrization of entire classes of linear phase multiband perfect reconstruction filter bank has remained open so far. This problem has recieved considerable attention in the 2-band case in [7]. In [1] an interesting solution to the problem of parametrizing all paraunitary LP filter banks is given. In [2] a subclass of biorthogonal LP are considered in the multiband situation. The problem of designing multiple band PR filter banks is known to be equivalent to the design of symmetric (or antisymmetric) wavelets. In this vein, the design of symmetric wavelets have been addressed in the work of [6] both in 2-band and in the multiple band case. In the present paper a method of parametrization and design of complete family of multiband LP, PR filter banks is presented by using techniques akin to Hermite reduction method of linear system theory. The essential idea behind the algorithm stems from the fact that in an M-band problem it is almost always possible to specify N ($N < M$) of the m analysis filters arbitarily and the remaining $M - N$ filters can be used to satisfy the PR property and desirable frequency characteristics as well.

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2 Biorthogonal linear phase filter bank

We assume in this section and in the rest of the paper that the indices (see the definition below) of the subband filters N_k are such that $N_k = Mn_k + l$; $0 \leq l \leq M-1$ for each k. We say the filter bank is of type l. In the $M = 2$ case, it can be shown that there does not exist any PR linear phase filter bank where $N_0 \bmod 2 \neq N_1 \bmod 2$. But the question of existence of PR linear phase filter banks when $N_k \bmod M$ are not the same for all $0 \leq k \leq M-1$ and $M > 2$ is open. We need the following definitions.

1. If a polynomial $Q(z)$ is such that $Q(z) = \pm z^n Q(z^{-1}) = \pm\tilde{Q}(z)$, then it is called self-(anti)symmetric polynomial with index n. Here, the index n of a self-(anti)symmetric polynomial $Q(z)$ is given by $n = \text{Deg}Q + q$, where q is the multiplicity of the root at $z = 0$ of $Q(z)$ and Deg$Q$ is the degree of $Q(z)$. We write $n = \text{Ind}(Q)$ for index of $Q(z)$

2. If two polynomials $R(z)$ and $P(z)$ are related by $R(z) = \pm z^m P(z^{-1}) = \pm\tilde{P}(z)$, then they are called cross-(anti)symmetric pair with index m. Here, the index m of cross-(anti)symmetric pair $R(z)$ and $P(z)$ is given by $m = (p + \text{Deg}R) = (r + \text{Deg}P)$, where p and r are multiplicity of the root at $z = 0$ of $P(z)$ and $R(z)$ respectively. We write $m = \text{Ind}(R, P)$ for index of the cross-(anti)symmetric pair R(z) and P(z), or simply $m = \text{Ind}(R)$ or $\text{Ind}(P)$ if there is no ambiguity.

Two row vectors, $\mathbf{h}'$ and $\mathbf{h}$ are:

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1. Structurally similar whenever $h_i = \pm\bar{h}_j$ if and only if $h'_i = \pm\bar{h}'_j$ or $h'_i = \mp\bar{h}'_j$

2. Index similar if, in addition to being structurally similar, they satisfy $\text{Ind}(h_i) - \text{Ind}(h_j) = \text{Ind}(h'_i) - \text{Ind}(h'_j)$ for all i and j.

where h_i and h'_i are the i-th elements of $\mathbf{h}$ and $\mathbf{h}'$ respectively. Next, we state useful facts for linear phase and perfect reconstruction filter banks, but all proofs are omitted for lack of space.

In a linear phase filter bank (LPFB), the polyphase components of each filter are related in the following way.

Fact 1 *Let $H_k(z)$ be the k-th filter of linear phase analysis filters in M-band filter bank, and let $H_k(z) = \bar{H}_k(z)$ (or $H_k(z) = -\bar{H}_k(z)$) with index $Mn_k + l$. Then, for each of its polyphase components $H_{k,i}(z)$, $i = 0, \cdots, M-1$, the following relation holds: $H_{k,i}(z) = \pm z^{n_k} H_{k,l-i}(z^{-1})$ for $0 \le i \le l$ and $H_{k,i}(z) = \pm z^{n_k - 1} H_{k,M+l-i}(z^{-1})$ for $l+1 \le i \le M-1$.*

Fact 2 *An analysis polyphase matrix corresponds to linear phase analysis filters bank if and only if any one of row is a valid LP row, i.e., its elements satisfy the relation given in Fact 1 and all other rows are index similar to this row.*

If the FIR filter bank is, in addition, the perfect reconstruction (PR), then the polyphase matrices corresponding to these filter banks have monomial determinants, and the converse is also true. In fact, we can find explicit structure of a LP polyphase matrix, but it is not given for lack of space. By using this and $\text{Det}\mathbf{H}(z) = z^L$, where $\mathbf{H}(z)$ is the analysis polyphase matrix, we obtain the following two results.

Fact 3 *If the analysis filters in an M-band PR filter bank satisfy the conditions for LP, then the number of symmetric and antisymmetric filters can be determined as follows:*

1. *Let M be odd. Then the number of symmetric filters exceed the number of antisymmetric filters by one.*

2. *Let M be even and $\text{Ind}(H_k) = Mn_k + l$; $0 \le l \le M-1$. Then*

   (a) if l is even, then the number of symmetric filters exceed the number of antisymmetric filters by two.

   (b) if l is odd, then there are equal number of symmetric and antisymmetric filters.

Fact 4 *In a linear phase, perfect reconstruction FIR filter bank $H_k(z)$; $0 \le k \le M-1$, where $N_k = \text{Ind}(H_k) = Mn_k + l$; $0 \le l \le M-1$, for each k, the following constraint holds.*

$$\sum_{i=0}^{M-1} N_k = M(2L + M - 1) \qquad (1)$$

We now describe the main results of this paper.

3 Linear phase reduction

Consider the situation when r analysis filters of the analysis polyphase matrix $\mathbf{H}(z)$ are given, i.e., the submatrix $\mathbf{H}_r^0$ of size $(r \times M)$ is given. Recall that all given filters are of the same type l, and thus, the rows of $\mathbf{H}_r^0$ are all index similar. Furthermore, the submatrix $\mathbf{H}_r^0$ must have a full rank for every $z \neq 0$.

Let us denote the i-th column of $\mathbf{H}_r^0$ by $\mathbf{h}_{m_i}^i$, where the subscript m_i represents the lowest index among the elements of $\mathbf{h}_{m_i}^i$. For convenience, we call m_i the index of the i-th column. Note that due to the structure of the LP polyphase matrix stated in Fact 2, $\min_j([\mathbf{h}^i]_j)$ is the same for all values of i, i.e., all elements having the smallest indices in columns always occur in the same row. We also use the following terminology:

1. A column vector is self-conjugate if its elements are self-symmetric or self-antisymmetric.

2. A pair of column vectors forms conjugate pair if their respective elements are either cross-symmetric or cross-antisymmetric.

From Fact 1, we easily see that each column pair $\{\mathbf{h}_{m_i}^i, \mathbf{h}_{m_{l-i}}^{l-i}\}$ for $0 \le i \le l$ and $\{\mathbf{h}_{m_i}^i, \mathbf{h}_{m_{M+l-i}}^{M+l-i}\}$ for $l+1 \le i \le M-1$ are conjugate. If l or $M+l$ is even, then the $\frac{l}{2}$-th or $\frac{M+l}{2}$-th column is self-conjugate.

Theorem 1 *Let $r < M$ filters of analysis bank be given. In other words, an $(r \times M)$ submatrix, $\mathbf{H}_r$ of the analysis polyphase matrix is given. Assuming that $\mathbf{H}_r$ is of rank r for all values of z except possibly $z = 0$, we can always find an $(M-r) \times M$ matrix $\mathbf{H}_{M-r}$ such that $\mathbf{H}^T = [\mathbf{H}_r^T \ \ \mathbf{H}_{M-r}^T]^T$ is a valid linear phase analysis polyphase matrix of an M-band PR filter bank.*

Conversely, if $\mathbf{H}_r$ is included as a submatrix of a linear phase analysis polyphase matrix, then $\text{rank}(\mathbf{H}_r) = r$ for all $z \neq 0$.

We propose a simple algorithm for this which uses two types of unimodular matrices called the transform matrices. However, when $r > 1$, the transform matrices may become singular which may become a some of a problem. Under the assumption that the inverse of transform matrices always exist in every reduction step, we show in the following that it is possible to construct a complete polyphase matrix $\mathbf{H}(z)$.

The i-th reduction step consists of the following

$$\mathbf{H}_r^{i-1} \cdot \mathbf{T}_{k_i}^i = \mathbf{H}_r^i \cdot \mathbf{S}_i \tag{2}$$

where $\mathbf{S}_i$ is a diagonal matrix of 1's and z's and $\mathbf{T}_{k_i}^i$'s are descibed below.

Transform matrix $\mathbf{T}_1^i$: Let a self-conjugate column of $\mathbf{H}_r^{i-1}$ have strictly larger index than that of any other pairs of conjugate columns. If l is even and the $\frac{l}{2}$-th column is such a self-conjugate column, we use transform matrix $\mathbf{T}_1^i$, the j-th column of which is given by

$$[\mathbf{T}_1^i]_j = \begin{cases} \mathbf{e}_j & \text{if } j \neq l/2 \\ \mathbf{t}_i & \text{if } j = l/2 \end{cases} \tag{3}$$

where $\mathbf{e}_j$ is the j-th column of $(M \times M)$ identity matrix. The j-th element of $\mathbf{t}_i$ is given as

$$[\mathbf{t}_i]_j = \begin{cases} t_{i,j} + z^{d_j} t_{i,l-j} & \text{if } j = 0, \cdots, l \\ t_{i,j} + z^{d_j} t_{i,M+l-j} & \text{if } j = l+1, \cdots, M-1 \end{cases} \tag{4}$$

where $d_j = m_{\frac{l}{2}} - m_j$. Let $\mathbf{H}_r^{i-1} \cdot \mathbf{T}_1^i = \mathbf{H}_r'^i$. Then, only two columns $\mathbf{h}_{m_{l/2}}^{i-1}$ and $\mathbf{h}_{m_{l/2}}'^i$ are different and other columns are exactly same. Furthermore, we can show that the j-th elements of $\mathbf{h}_{m_{l/2}}^{i-1}$ and $\mathbf{h}_{m_{l/2}}'^i$ have the same index and the same symmetry. Let us choose free parameters $t_{i,j}$, $j = 0, \cdots, M-1$, in $\mathbf{h}_{m_{l/2}}'^i$ to satisfy the relation.

$$\mathbf{H}_r^{i-1} \cdot \mathbf{t}_i = \mathbf{0} \quad \text{for } z = 0 \tag{5}$$

Here, we assume that the inverse of the transform matrix $\mathbf{T}_1^i$ exists, i.e., $\mathrm{Det}\mathbf{T}_1^i = 2t_{i,\frac{l}{2}} \neq 0$. If (5) is satisfied, then every element of $\mathbf{h}_{m_{l/2}}'^i$ has a zero at $z = 0$ and simultaneously every leading term becomes zero by its property of self-symmetry. Thus, $\mathbf{H}_r'^i$ can be factored into $\mathbf{H}_r^i \cdot \mathbf{S}_i$, where the diagonal matrix $\mathbf{S}_i$ has a z at the $l/2$-th entry and 1's elsewhere.

Transform matrix $\mathbf{T}_2^i$: If a pair of conjugate columns of $\mathbf{H}_r^{i-1}$ in (2) has index larger than or equal to the index of any other column, then we use transform matrix $\mathbf{T}_2^i$.

When the k-th and the $(l-k)$-th columns form such a column pair, where $0 \leq k \leq l$, we use the transform matrix $\mathbf{T}_2^i$, the j-th column of which is given by

$$[\mathbf{T}_2^i]_j = \begin{cases} \mathbf{e}_j & \text{if } j \neq k,\ j \neq l-k \\ \mathbf{t}_i & \text{if } j = k \\ \mathbf{t}_i' & \text{if } j = l-k \end{cases} \tag{6}$$

The j-th element of $\mathbf{t}_i$ is given by

$$[\mathbf{t}_i]_j = t_{i,j}(1 + z^{d_j}), \quad \mathbf{t}_i' = \mathrm{Diag}[\mathbf{J}_{l+1} \mid \mathbf{J}_{M-l-1}] \cdot \mathbf{t}_i \tag{7}$$

where $\mathbf{J}_n$ is the skew identity matrix of size $n \times n$ and $d_j = m_k - m_j = m_{l-k} - m_j$. Let $\mathbf{H}_r^{i-1} \cdot \mathbf{T}_2^i = \mathbf{H}_r'^i$. Then, we can show that the column pair $\{\mathbf{h}_{m_k}'^i, \mathbf{h}_{m_{l-k}}'^i\}$ is also conjugate and has the same index as the pair $\{\mathbf{h}_{m_k}^{i-1}, \mathbf{h}_{m_{l-k}}^{i-1}\}$. Next, we choose free parameters $t_{i,j}$ in $\mathbf{h}_{m_k}'^i$ or $\mathbf{h}_{m_{l-k}}'^i$ so that one of the following equations is satisfied.

$$\mathbf{H}_r^{i-1} \cdot \mathbf{t}_i = 0 \quad \text{or} \quad \mathbf{H}_r^{i-1} \cdot \mathbf{t}_i' = 0 \quad \text{for } z = 0 \tag{8}$$

Here, we assume that the inverse of $\mathbf{T}_2^i$ exists, i.e., $t_{i,k}^2 \neq t_{i,l-k}^2$. If the first one of equations (8) is satisfied, then every element of $\mathbf{h}_{m_k}'^i$ has a zero at $z = 0$ and simultanuously every leading term of elements in $\mathbf{h}_{m_{l-k}}'^i$ becomes zero by the property of cross-symmetry. Thus, the matrix $\mathbf{H}_r'^i$ can be factored into $\mathbf{H}_r^i \cdot \mathbf{S}_i$, where the diagonal matrix $\mathbf{S}_i$ has a z at the k-th position and 1's at other positions.

If the second equation in (8) is satisfied, then the diagonal matrix $\mathbf{S}_i$ has a z at the $(l-k)$-th position and 1's at other positions.

3.1 Construction of LP and PR filter bank

In this section, we will give a method of constructing the entire polyphase matrix, thus, an analysis filter bank, satisfying the condition for linear phase as well as the condition for perfect reconstruction form $\mathbf{H}_r$.

Inverse transform matrix: $\mathbf{S}_i \cdot \mathbf{T}_{k_i}^i$

We construct $\mathbf{T}_{k_i}^i$ so that the inverse matrix $\mathbf{S}_i \cdot \mathbf{T}_{k_i}^i$ has the following properties.

1. Since $\mathbf{T}_{k_i}^i$ is a unimodular, the determinant of the inverse matrix is a monomial in z.

2. The inverse matrix has structure invariant property, which is defined below

Definition 1 *We consider a class of square matrices* $\mathbf{T}$ *that right multiplies index similar row vectors: The matrix* $\mathbf{T}$ *is said to be structure invariant if*

1. $\mathbf{h}' = \mathbf{h} \cdot \mathbf{T}$ *and* $\mathbf{h}$ *are structurally similar.*

2. If $\mathbf{h}_1$ and $\mathbf{h}_2$ are two index similar rows, then $\mathbf{h}_1 \cdot \mathbf{T}$ and $\mathbf{h}_2 \cdot \mathbf{T}$ are index similar for any $\mathbf{h}_1$ and $\mathbf{h}_2$.

Claim 1 *The inverse transform matrix $\mathbf{S}_i \cdot (\mathbf{T}^i_{k_i})^{-1}$ has the structure invariant property.*

Construction of polyphase matrix

By reduction method descibed above, we obtain a matrix $\mathbf{H}^N_r$ having the lowest column index, which is given by

$$\mathbf{H}^0_r = \mathbf{H}^N_r \cdot \mathbf{T}$$

Since similarity is a transitive property, it easily follows that $\mathbf{T}$ is a structure invariant matrix. Now, consider a matrix $\mathbf{F}$ whose first r rows are $\mathbf{H}^N_r$. Since $\mathbf{T}$ is a structure invariant matrix, if we choose the other $M-r$ rows of $\mathbf{F}$ so that they are index similar to $\mathbf{H}^N_r$, then $\mathbf{F} \cdot \mathbf{T}$, whose first r rows are $\mathbf{H}^0_r$, has all index similar rows to $\mathbf{H}^0_r$ by the property of structure invariance. Then, since every row of $\mathbf{H}^0_r$ is a valid LP row, by Fact 2, $\mathbf{F} \cdot \mathbf{T}$ is a desired LP polyphase matrix. However, that all rows of $\mathbf{F} \cdot \mathbf{T}$ are index similar to $\mathbf{h}^0$ is not enough for PR. For PR, the determinant of $\mathbf{F} \cdot \mathbf{T}$ must be a monomial in z. Since the determinant of $\mathbf{T}$ is a monomial, we need that the determinant of $\mathbf{F}$ be a monomial. Thus, we select $\mathbf{F}$ so that the last $M-r$ rows are index similar to the first r rows and its determinant is a monomial in z. In fact, $\mathbf{F}$ must be the polyphase matrix of a LP filter bank of low degree. The construction of this from the first r rows can be conveniently accomplished by using Facts 3 and 4.

Example: We consider $M = 4$. The specified analysis filters are

$$H_0(z) = .001 - .0059z - .0218z^2 - .0079z^3 + .0981z^4 + .2645z^5 + .3486z^6 + .2645z^7 + .0981z^8 - .0079z^9 - .0218z^{10} - .0059z^{11} + .001z^{12}$$
$$H_1(z) = .0036 + .0195z - .0265z^2 - .2709z^3 + .5508z^4 - .2709z^5 - .0265z^6 + .0195z^7 + .0036z^8$$

By following the design algorithm described above, we obtain $H_2(z)$ and $H_3(z)$. The frequency responses of these 4 filters are shown in Fig. 1. While the filters $H_0(z)$ and $H_1(z)$ were designed independently of the PR requirement, low degree $H_2(z)$ and $H_3(z)$ were first obtained by the algorithm descibed above. In order to achieve the bandpass and highpass characteristics, higher degree solution were obtained from these. Details are available in [8].

4 Conclusions

We have presented a complete algorithmic method for the construction of LP, PR filter banks from a incomplete set of arbitrarily specified subset of the filters of the analysis bank. The algorithm is reminiscent of Hermite reduction in system theory and can also be shown to provide [8] a cascade-like decomposition of a completely specified analysis bank, thus, providing a complete parametrization of all M-band LP, PR filter banks.

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Fig 1. The magnitude responses of analysis filters in Example