LINEAR-PHASE M-BAND WAVELETS WITH APPLICATION TO IMAGE CODING

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ABSTRACT

This paper investigates the design of M-band linear phase wavelet filter banks ($M > 2$), and explores their application to image coding. The generalized LOT description of M-band linear-phase paraunitary filter banks is used to parametrize the M-band linear-phase orthogonal wavelets. It is proven that an M-band linear-phase orthogonal wavelet of even length *cannot* have more than one vanishing moment. Since this limits the effectiveness of the resulting wavelet filters, we next suggest methods for the construction of linear-phase biorthogonal M-band wavelet lowpass filters, generalizing prior 2-band constructions. However, one cannot guarantee that an arbitrary lowpass filter pair can be completed to a full perfect-reconstruction filter bank. Finally, the new linear-phase orthogonal wavelet filter banks are compared with known wavelet filters with regard to their performance in a transform-based image coder.

1. INTRODUCTION

Discrete wavelets have recently drawn attention as a multiresolution transform; they are especially well-suited to image coding because they match perceptual properties of the human visual system. However, 2-band wavelets cannot be simultaneously orthogonal and symmetric (linear-phase). Linear-phase filter banks are desirable for subband coders because they enable symmetric extension at image boundaries and they preserve centers of mass in an iterated decomposition such as the wavelet transform. In the 2-band case, this obstacle has been overcome [8, 1, 14] by designing biorthogonal (perfect-reconstruction, not paraunitary) filters. However, when $M > 2$, linear-phase and orthogonality can be simultaneously satisfied. Such Orthogonal linear-phase filter banks have recently been parametrized [12, 10], at least when M is even. We use these parametrizations to determine the M-band orthogonal linear-phase wavelets.

Regularity, often measured by the number of vanishing moments, is a key property of a wavelet filter bank; it determines the smoothness of the iterated lowpass filter. We prove that an M-band orthogonal linear-phase filter bank with even-length lowpass filters *cannot* have two or more vanishing moments, and thus the associated scaling functions cannot even have a continuous derivative. Given this mathematical obstruction, one must turn to odd-length filters or biorthogonal systems to obtain M-band symmetric wavelets with arbitrary regularity. We generalize the 2-band biorthogonal constructions of Daubechies and Vetterli-Herley to the M-band case, devising linear-phase lowpass wavelet filters with an arbitrary number of vanishing moments. Thus we are able to create M-band symmetric scaling functions with any desired smoothness. However, there is no guarantee that an arbitrary lowpass filter pair can be completed to a perfect-reconstruction filter bank.

We employ the new M-band orthogonal wavelet filter banks in an image compression system. The wavelets are used to decompose the image into a set of subbands. Entropy-constrained scalar quantization is then performed on each subband, using operational rate-distortion methods [11] to determine the optimal bit allocation. Finally, we compare the performance of the new filter banks with Daubechies' wavelets.

2. LINEAR-PHASE ORTHONORMAL M-BAND WAVELETS

M-band orthonormal filter banks in which each filter has linear-phase symmetry have recently been parametrized [12, 10]. The parametrization of [10] establishes such filter banks as generalizations of the lapped orthogonal transform (LOT) [7], and uses the name GenLOT. Let us summarize this parametrization briefly. An M-band filter bank can be described in the

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z-transform domain by its polyphase matrix $\mathbf{E}(z)$:

$$\begin{bmatrix} \mathbf{H}_0(z) \\ \mathbf{H}_1(z) \\ \vdots \\ \mathbf{H}_{M-1}(z) \end{bmatrix} = \mathbf{E}(z) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix} .$$

When the number of channels in the filter bank M is even, and the filters $h_k[n]$ (with z-transforms $\mathbf{H}_k(z)$) have length MN, the polyphase matrix has the form

$$\mathbf{E}(z) = \mathbf{K}_{N-1}(z)\mathbf{K}_{N-2}(z)\ldots\mathbf{K}_1(z)\mathbf{E}_0 \qquad (1)$$

where each

$$\mathbf{K}_i(z) = \frac{1}{2}\begin{bmatrix} \mathbf{U}_i & 0 \\ 0 & \mathbf{V}_i \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & 0 \\ 0 & z^{-1}\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} .$$

$\mathbf{I}$ is the rank $M/2$ identity matrix, while $\mathbf{U}_i$ and $\mathbf{V}_i$ are arbitrary rank $M/2$ orthogonal matrices. $\mathbf{E}_0$ is a rank M unitary matrix with $M/2$ symmetric and $M/2$ antisymmetric rows (such as the DCT-IV); it can be factored as

$$\mathbf{E}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{D}_0 & 0 \\ 0 & \mathbf{D}_1 \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{J} \end{bmatrix} ;$$

$\mathbf{J}$ is the familiar reverse identity matrix (of rank $M/2$), while $\mathbf{D}_0$ and $\mathbf{D}_1$ are arbitrary orthogonal matrices of rank $M/2$.

We examine the use of the GenLOT in the construction of M-band linear-phase orthonormal wavelets, and obtain two interesting results. First, all possible GenLOTs with one vanishing moment may be described in terms of certain rotation matrices. An orthogonal wavelet filter bank has one vanishing moment [13, 15] if it satisfies

$$\begin{bmatrix} \mathbf{H}_0(z) \\ \mathbf{H}_1(z) \\ \vdots \\ \mathbf{H}_{M-1}(z) \end{bmatrix} = \begin{bmatrix} \sqrt{M} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (2)$$

for $z = 1$. This means that constant signals are "pure lowpass," yielding zero outputs from the bandpass and highpass filters. This condition is equivalent to the existence of a corresponding tight frame for $L^2(\mathbf{R})$ [13]. We find that a GenLOT with $N-1$ factors $\mathbf{K}_i(z)$ will have one vanishing moment if and only if the rotation matrices $\mathbf{U}_i$ satisfy

$$\mathbf{U}_{N-1}\mathbf{U}_{N-2}\ldots\mathbf{U}_1\mathbf{D}_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{M/2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

Clearly, the number of free parameters available for filter design (to optimize such properties as stopband attenuation or coding gain) increases with both the number of channels M and the filter length, which is determined by the number of factors N. With even the simplest $N = 1$ LOTs, it is possible to create a filter bank based on the DCT-IV having one vanishing moment for use in a wavelet decomposition. A 4-band GenLOT with $N = 2$ (filter length 12) and one vanishing moment is shown in Figure 1.

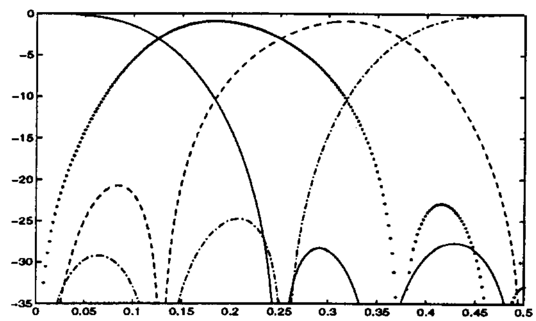

Figure 1: Magnitude responses (in dB) of a 4-channel GenLOT with one vanishing moment. Each filter has 12 taps.

It is of interest to create wavelet filters with more than one vanishing moment; indeed, the interpolation/approximation properties of such filters lead to their superiority for wavelet-based image coding systems [5]. Daubechies discovered orthonormal wavelets with N vanishing moments in the 2-channel case [2]; her construction was generalized to the M-channel case in [13, 15] . An M-channel orthogonal filter bank will have two vanishing moments if it satisfies (2) as well as the second-order condition

$$\frac{d}{dz}\begin{bmatrix} \mathbf{H}_0(z) \\ \mathbf{H}_1(z) \\ \vdots \\ \mathbf{H}_{M-1}(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (3)$$

at $z = 1$. However, we have the following negative result:

Theorem 2.1 *An M-channel linear-phase orthogonal filter bank with lowpass filters of even length L cannot have more than one vanishing moment, independent of M and L.*

Proof: The one-vanishing-moment condition (2) implies

$$\sum_{n=0}^{L-1} h_0[n] = \sqrt{M} \Leftrightarrow \sum_{n=0}^{L/2-1} h_0[n] = \sqrt{M}/2 , \qquad (4)$$

for an even-length symmetric filter. On the other hand, combining the second-vanishing-moment condition (3) with the half-sample symmetry of the filter implies

$$\sum_{n=0}^{L-1} n h_0[n] = 0 \Leftrightarrow \sum_{n=0}^{L/2-1} h_0[n] = 0 \ . \qquad (5)$$

The two equations (4) and (5) cannot simultaneously be true; this obstruction cannot be resolved by increasing the order of the filters involved.

As a corollary of this result, the scaling functions associated with an even-length symmetric filter pair may be continuous, but they cannot have even one continuous derivative [2]. Theorem 2.1 is supported by several facts. The only linear-phase 2-channel orthogonal wavelet filter bank is the Haar system, which has one vanishing moment. Second, the authors do not know of a published example of an M-channel linear-phase orthogonal filter bank with two or more vanishing moments. It may be possible to create linear-phase M-channel orthogonal wavelets of *odd* length with more than one vanishing moment.

3. LINEAR-PHASE BIORTHOGONAL M-BAND WAVELETS

Given the result of the previous section, and the desirability of more than one vanishing moment in a wavelet filter bank, we now examine the possibility of constructing linear-phase M-band wavelet filter banks with N vanishing moments that are biorthogonal rather than orthogonal. This corresponds to perfect-reconstruction rather than paraunitarity filter banks. In the 2-band case [1, 14], this approach was successful. The idea is to factor the modulus squared $|\mathbf{A}_0(z)|^2$ of an orthogonal lowpass wavelet filter with N vanishing moments to yield two symmetric filters $\mathbf{H}_0$ and $\mathbf{F}_0$ of different lengths such that

$$\mathbf{H}_0(z)\overline{\mathbf{F}_0(z)} = |\mathbf{A}_0(z)|^2 \ .$$

Each of $\mathbf{H}_0$ and $\mathbf{F}_0$ will have some integer number of vanishing moments.

In the M-band case, closed-form solutions for orthogonal wavelet filters with N vanishing moments (but not linear phase symmetry!) have been obtained [13]. Thus we have $|\mathbf{A}_0(z)|^2$ at our disposal and can split it into linear-phase factors. Care is required to obtain filters with well-behaved frequency responses, as well as smooth scaling functions for the underlying basis of $L^2(\mathbf{R})$. One such example for $M = 4$ with an analysis filter having 2 vanishing moments and a synthesis filter having 4 vanishing moments is shown in Figure 2. This example was obtained by redistributing the roots of the modulus squared of the 4-band lowpass filter with 3 vanishing moments and an additional zero at $3\pi/4$ described in [4].

Given a pair of lowpass M-band wavelet filters, one would like to obtain bandpass and highpass filters which yield a linear-phase M-band perfect-reconstruction filter bank. This does not appear to be possible for the (11,19)-tap pair described above, and points out the importance of deriving lattice-structure-based parametrizations of M-band linear-phase filter banks, such as those in [8, 9]. By working with such a factorization, one would hope to simultaneously obtain a linear-phase perfect-reconstruction filter bank, and wavelets with more than one vanishing moment.

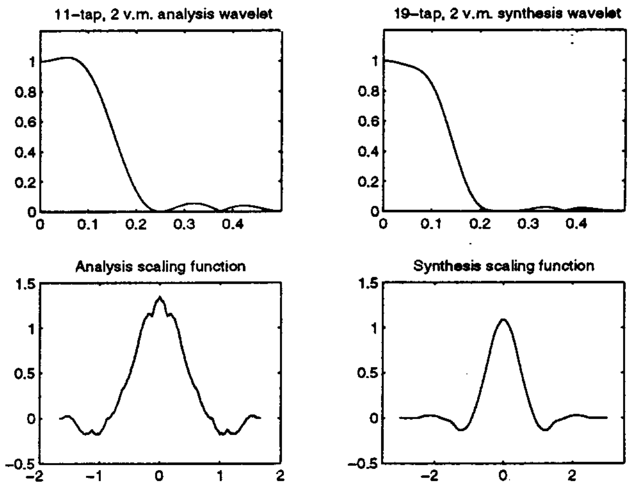

Figure 2: Magnitude responses and scaling functions for a 4-channel biorthogonal wavelet lowpass filter pair.

4. APPLICATION TO IMAGE CODING

Finally, having constructed a new family of M-channel linear-phase wavelet filter banks, we evaluate their effectiveness for image coding on a variety of images. The linear-phase wavelet filter banks serve as the transform element in a transform coding system. The filter banks are used to decompose the image into a set of critically sampled subbands; the wavelet decomposition should provide superior energy compaction in this subband decomposition. We then apply entropy-constrained uniform scalar quantization and entropy coding to the subband data. Wavelet subband data is well-modeled by Laplacian distributions [6]. We use this model to determine an optimal bit allocation among the subbands, using the operational rate-distortion methodology of [11]. We employ a combination of Huffman and zero-run-length encoding to approach the true entropy of the quantized bitstream.

We applied this coding system to the Lenna image (NITF6), an 8-bit fingerprint image, and a 2-dimensional seismic dataset. Our compression results (based on the size of the entropy-coded bitstream) are shown in Tables 1, 2, and 3. For the seismic dataset, we compared a 2-channel, 5-level Mallat tree based on the Daubechies (7,9)-tap biorthogonal filter pair [1] with a

4-channel, 3-level Mallat tree [6] based on the 12-tap GenLOT shown in Figure 1. The 4-channel wavelet offered superior or comparable performance across all compression ratios. We employed the same wavelet transforms on the Lenna image, and found that at a compression ratio of 8:1, the 4-channel wavelet system had a lower maximum error than the 2-channel wavelet, and was within 0.5 dB in pSNR. For the fingerprint image, we compared two discrete wavelet transforms. One of them was the transform of the FBI's Wavelet Scalar Quantization standard [3], which uses the (7,9)-tap filter in a specific non-Mallat tree. We compared this transform with a mixed 4-band and 2-band tree which achieves an identical subband decomposition of the image by cascading 2 levels of the 4-band GenLOT and then applying a 2-band filter to the lowest-frequency subblock. In this case, the 4-channel filters yielded superior maximum error and slightly lower pSNR's than the 2-channel (7,9)-tap transform, at compressions of 16:1 and 32:1. The effectiveness of the 4-channel GenLOT is remarkable, considering that it has only one vanishing moment, whereas the (7,9)-tap filter pair has four vanishing moments! It is hoped that further work on filter design (such as biorthogonal pairs with more vanishing moments) will yield superior transform coder performance.

	8:1		16:1		32:1	
	pSNR	Max	pSNR	Max	pSNR	Max
$M = 2$	51.6	1101	40.7	4262	32.7	12544
$M = 4$	52.6	1025	41.3	4298	33.7	12260

Table 1: Peak SNR and maximum errors for compression of 2-d seismic data example (Mallat tree, $M = 2$ (7,9)-tap pair and $M = 4$ GenLOT).

	8:1		16:1		32:1	
	pSNR	Max	pSNR	Max	pSNR	Max
$M = 2$	37.8	27	34.8	40	31.7	59
$M = 4$	37.4	25	33.8	46	30.3	64

Table 2: Peak SNR and maximum errors for compression of *Lenna* (Mallat tree, $M = 2$ (7,9)-tap pair and $M = 4$ GenLOT).

	8:1		16:1		32:1	
	pSNR	Max	pSNR	Max	pSNR	Max
$M = 2$	36.5	18	31.8	34	28.2	87
$M = 4$	35.6	22	31.0	33	27.7	74

Table 3: Peak SNR and maximum errors for compression of fingerprint image (WSQ tree, $M = 2$ (7,9)-tap pair and $M = 4$ GenLOT).

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