LATTICE STRUCTURE FOR TWO-BAND PERFECT RECONSTRUCTION FILTER BANKS USING PADÉ APPROXIMATION*

Masoud R.K. Khansari
Department of EECS
University of California
231 Cory Hall
Berkeley, CA, 94720
e-mail: masoud@diva.eecs.berkeley.edu

Eric Dubois
INRS-Telecommunications
16 Place du Commerce
Nuns' Island, Verdun
Quebec, Canada, H3E 1H6
e-mail: eric@inrs-telecom.uquebec.ca

ABSTRACT

We show how the Padé table can be utilized to develop a new lattice structure for general two-channel bi-orthogonal perfect reconstruction (PR) filter banks. This is achieved through characterization of *all* two-channel bi-orthogonal PR filter banks. The parameter space found using this method is unique for each filter bank. Similarly to any other lattice structure, the PR property is achieved structurally and quantization of the parameters of the lattice does not effect this property. Furthermore, we demonstrate that for a given filter, the set of all complementary filters can be uniquely specified by two parameters, namely the end-to-end delay of the system and a scalar quantity.

1. INTRODUCTION

Factorization of filter banks has proved to be essential in both design and implementation of PR filter banks. Also, they help in understanding the fundamental algebraic structure of PR filter banks. Using factorization, a higher order filter bank is achieved by adding an extra element to a lower order bank. Such a lattice type structure was considered to characterize all M-channel paraunitary filter banks [1]. Partial results on factorization of general PR filter banks have been recently reported where, although the proposed factorization characterizes a large class of PR filter banks, it lacks completeness [2]. Such structures have also been proposed for the bi-orthogonal linear-phase filters [3]. With the use of ladder structures, it can be shown that the factorization of general M-channel bi-orthogonal PR filter banks is possible. This factorization even though is complete lacks uniqueness [4].

In [5] Vetterli and Herley demonstrated the close relation between the continued fraction expansion (CFE) of functions and PR filter banks. They showed that in the case of two-band PR filter banks with filters $H(z)$ and $\widetilde{H}(z)$, the CFE of the ratios of $H(z)/H(-z)$ and $\widetilde{H}(z)/\widetilde{H}(-z)$ are similar except for the last term. This is expected in light of the fact that the comprimeness of $H(z)$ and $H(-z)$ is a necessary condition for PR, which also points to the relation between the CFE and the Euclid algorithm as was demonstrated in [5]. The Padé table is a classical method for approximating a power series by a ratio of two polynomials [6]. The members of the table approximate a given power series as closely as possible and are distinguished by the respective orders of their numerators and denominators. Clearly, the higher the order of these polynomials, the closer is the approximation to the given power series. Also, any path [1] through the Padé table of a function corresponds to a CFE of that function. However, the converse is not true and not every CFE of a function corresponds to a path of its Padé table [6].

In Section 2, we first give a brief summary of the Padé approximation and the CFE and elaborate on their properties which are directly used in this paper. We then develop a new lattice structure for general two-channel bi-orthogonal PR filter banks in Section 3. This is done through use of the properties of the Padé table and the CFE and characterization of all two-channel bi-orthogonal PR filter banks. The PR property is achieved structurally and quantization of the lattice parameter does not effect this property. In Section 4, we develop a two-parameter characterization of all the possible complementary filters of a given filter where one parameter is related to the end-to-end delay of the system and the other is a scalar quantity.

2. PADÉ TABLE AND CONTINUED FRACTION EXPANSION

The $[m, n]$ (mth row, nth column) element of the Padé table of function $f(z)$ is the ratio of two relatively prime polynomials in z, namely $P_{m,n}(z)/Q_{m,n}(z)$ where the highest degree of $P_{m,n}(z)$ and $Q_{m,n}(z)$ are at most m and n, respectively [6]. If these degrees are exactly equal to m and n for all the elements of the table, then the function $f(z)$ is called *regular*. The elements of the Padé table are defined such that,

$$f(z) - \frac{P_{m,n}(z)}{Q_{m,n}(z)} = \mathcal{O}(z^{m+n+1}) \tag{1}$$

where $\mathcal{O}(z^l)$ denotes a power series with elements of degree l and higher. Utilizing the above, one can show that

$$[n, n] - [n-1, n-1] = \frac{\alpha z^{2n-1}}{Q_{n,n}(z)Q_{n-1,n-1}(z)}, \tag{2}$$

which is the expression for the difference between successive approximations of $f(z)$ along the diagonal elements of the

* This research was supported in part by the National Sciences and Engineering Research Council (NSERC) of Canada.

[1] By path we mean to start from upper left corner of the table, and at each step, to move to the right or the down or along the diagonal direction.

0-7803-2431-5/95 $4.00 © 1995 IEEE

table. Throughout, we take advantage of the following two invariance properties of the Padé table:

- If $P(z)/Q(z)$ is the $[m,n]$ element of the Padé table for $f(z)$, then $Q(z)/P(z)$ is the $[n,m]$ element of the Padé table for $1/f(z)$.
- If $P(z)/Q(z)$ is the $[n,n]$ element of the Padé table for $f(z)$, then $P(-z)/Q(-z)$ is the $[n,n]$ element of the Padé table for $f(-z)$.

The nth approximant of the continued fraction

$$b_0(z)+\frac{a_1(z)}{b_1(z)+}\,\frac{a_2(z)}{b_2(z)+\cdots} \triangleq b_0(z)+\frac{a_1(z)}{b_1(z)+\frac{a_2(z)}{b_2(z)+\cdots}}, \quad (3)$$

is defined as

$$\frac{A_n(z)}{B_n(z)} = b_0(z)+\frac{a_1(z)}{b_1(z)+}\,\frac{a_2(z)}{b_2(z)+\cdots}\,\frac{a_n(z)}{b_n(z)}, \quad (4)$$

where it can be shown that both the numerators ($A_n(z)$) and the denominators ($B_n(z)$) satisfy the same *three-term recurrence* equation [7]

$$\begin{cases} A_n(z) &= b_n(z)A_{n-1}(z)+a_n(z)A_{n-2}(z) \\ B_n(z) &= b_n(z)B_{n-1}(z)+a_n(z)B_{n-2}(z) \end{cases} \quad (5)$$

but with different initial conditions

$$\begin{cases} A_0(z)=b_0(z) & A_{-1}(z)=1 \\ B_0(z)=1 & B_{-1}(z)=0 \end{cases} . \quad (6)$$

The CFE corresponding to the path passing through the diagonal elements of the Padé table is known as P-fraction expansion where $a_n(z)=1$ for all n and $b_n(z)$ are polynomials of z^{-1} [8] [2]. In the case of regular $f(z)$, it is straightforward to show that $b_n(z)$ is at most a first order polynomial. For example, the P-fraction expansion of $e^z=\sum_{k=0}^{\infty} z^k/k!$ is given by

$$\begin{aligned} e^z &= 1+\frac{1}{z^{-1}-\frac{1}{2}+}\,\frac{1}{12z^{-1}+}\,\frac{1}{5z^{-1}+\ldots} \\ &= 1+\frac{z}{1-\frac{1}{2}z+}\,\frac{z^2}{12+}\,\frac{z^2}{5+\ldots} \end{aligned}$$

3. LATTICE STRUCTURE

We first prove the following theorem

Theorem 1 *If $f(z)f(-z)=1$, then the numerators of the consecutive diagonal elements of the Padé table of $f(z)$ are a PR pair.*

Proof: Since $f(z)^{-1}=f(-z)$, then from the invariance properties of the Padé table, one can show that

$$\frac{P(z)}{Q(z)}=\frac{Q(-z)}{P(-z)}$$

where $[n,n]=P(z)/Q(z)$ is the nth diagonal of the Padé table of $f(z)$. Now since $P(z)$ and $Q(z)$ are relatively prime, we have $Q(z)=P(-z)$. As a result, $[n,n]$ is of the form $H_n(z)/H_n(-z)$ and applying (2), we arrive at

$$\frac{H_n(z)}{H_n(-z)}-\frac{H_{n-1}(z)}{H_{n-1}(-z)}=\frac{\alpha z^{2n-1}}{H_n(-z)H_{n-1}(-z)},$$

or

$$H_n(z)H_{n-1}(-z)-H_n(-z)H_{n-1}(z)=\alpha z^{2n-1}$$

which means that $\{H_n(z),H_{n-1}(z)\}$ is a PR pair.□

[2] Note that $f(z)$ is a power series in z. In this paper, we denote the delay element by z instead of z^{-1}.

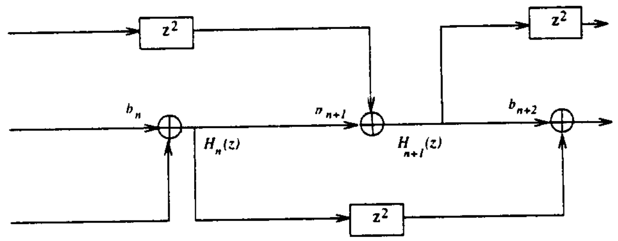

Figure 1: Modular implementation of successive approximants

Since the diagonal elements of the Padé table can be also found through P-fraction expansion of $f(z)$, the numerators of the diagonal elements of the table satisfy the three-term recurrence given by equation (5). Moreover, if $f(z)f(-z)=1$, one can show that $b_n(z)=b_nz^{-1}$ for $n\geq 2$ and

$$\begin{cases} H_1(z) &= (b_1-\frac{1}{2}z)H_0(z)+zH_{-1}(z) \\ H_n(z) &= b_nH_{n-1}(z)+z^2H_{n-2}(z) \qquad n>1, \end{cases} \quad (7)$$

where $b_n\in\mathbb{C}$, $H_0(z)=H_{-1}(z)=1$ and as before $H_n(z)$ is the numerator of the nth diagonal element of the Padé table. In summary, as shown in Figure 1, the nth order filter $H_n(z)$ is found by shifting $H_{n-2}(z)$ by 2 and then adding an appropriate multiple of $H_{n-1}(z)$. Note that from Theorem (1), the pair $\{H_{n-1}(z),H_n(z)\}$ is a PR filter bank for all n. Also, the circuit shown in Figure (1) is structurally PR and this property is not effected by the choice or the quantization of b_n.

Table (1) shows Daubechies D4 filter and its first two approximants where the coefficients are normalized so that the leading coefficient is one. Note that from the above theorem the pairs $\{H_7(z),H_6(z)\}$ and $\{H_6(z),H_5(z)\}$ are both PR pair. Table (2) shows the value of b_i ($1\leq i\leq 7$) which results in Daubechies D4 filter.

Remark 1: Note that, for a given $H_n(z)$, the ordered sequence $\{b_1,\cdots,b_n\}$ found through this procedure is unique and it is not possible to find another sequence $\{b_1',\cdots,b_n'\}$ to arrive at the same $H_n(z)$.

Remark 2: It is straightforward to verify the following lemma.

Lemma 1 *If $H(z)$ and $\widetilde{H}(z)$ are of order n and $n-1$ respectively and are a PR pair, then any other filter of order $n-1$ which forms a PR pair with $H(z)$ is of the form $\delta\widetilde{H}(z)$ for some $\delta\in\mathbb{C}$.*

In other words, the filter of order $n-1$ which forms a PR pair with $H_n(z)$ is unique up to a scaling factor. Therefore, it is sufficient to use any method such as Euclid Algorithm instead of Padé table to find $H_{n-1}(z)$.

Remark 3: It can be shown that if $\lim b_n \to \infty$ then $\lim_{n\to\infty} H_n(z)$ exists for all $z \in \mathbb{C}$. This fact can be utilized to find a sequence of filters $\{H_n(z)\}$ such that they approach a specific frequency response and any two consecutive filters constitute a PR filter bank.

4. COMPLEMENTARY FILTER

From Remark 3 of the last section, we can see that the frequency response of $H_n(z)$ and $H_{n-1}(z)$ can be close to each other. In practice, however, one wants the complementary filter to have high-pass response to capture those frequency components that are not significantly present in the low-pass filter. The following theorem shows that *all* the order n complementary filters $\widetilde{H}_n(z)$ can be found using a combination of $H_{n-1}(z)$ and $H_n(z)$. Moreover, if we restrict $\widetilde{H}(z)$ to be causal, only two parameters (up to a scaling factor) are sufficient to characterize $\widetilde{H}(z)$.

Theorem 2 *Let $H_n(z)$ and $H_{n-1}(z)$ of order n and $n-1$, respectively, be a PR pair. Then any $\widetilde{H}(z)$, of order n, such that $H_n(z)$ and $\widetilde{H}(z)$ are a PR pair with*

$$H_n(z)\widetilde{H}(-z) - \widetilde{H}(z)H_n(-z) = \alpha z^{(2k+1)},$$

where $k \in \{0, 1, \cdots, n-1\}$, has the following form:

$$\widetilde{H}(z) = \delta z^{-2(n-k-1)} H_{n-1}(z) + \left(\sum_{j=0}^{n-k-1} \beta_{j+1} z^{-2j}\right) H_n(z) \tag{8}$$

for some $\beta_1, \cdots, \beta_{n-k}, \delta \in \mathbb{C}$. Moreover, if k, δ and β_1 are specified, only one choice of $\beta_2, \cdots, \beta_{n-k}$ yields a causal filter for $\widetilde{H}(z)$.

Proof: The proof is given in the Appendix. □

Note that for a given value of k, the frequency selectivity of $\widetilde{H}(z)$ depends only on the ratio δ/β_1 and not their absolute values. Therefore, for a given k, the process of finding a complementary filter $\widetilde{H}(z)$ with a desired frequency response is a *one-dimensional* search through different values of δ/β_1. Also note that k is related to the end-to-end delay of the system which is equal to $2k+1$ and hence small value of k corresponds to low-delay PR filter bank. From Theorem (2) we conclude that any causal complementary filter of order n of $H_n(z)$ is uniquely characterized (up to a scaling factor) by the two parameter set $\{k, \delta/\beta_1\}$ or there are only two degrees of freedom in choosing this filter.

Figure (2) shows the frequency response of $\widetilde{H}(z)$ where $k=7$ and δ/β_1 is varied from -95 to -85. The low-pass filter prototype used in this figure is the Daubechies D4 ($H_7(z)$ in Table (1)). Note that around $\delta/\beta_1 = -90$, $\widetilde{H}(z)$ has the desired high-pass frequency response. Indeed, one can show that $\{k=7, \delta/\beta_1 = -90.5\}$ corresponds to power complementary filter of $H(z)$, namely $\widetilde{H}(z) = z^7 H_7(-z^{-1})$.

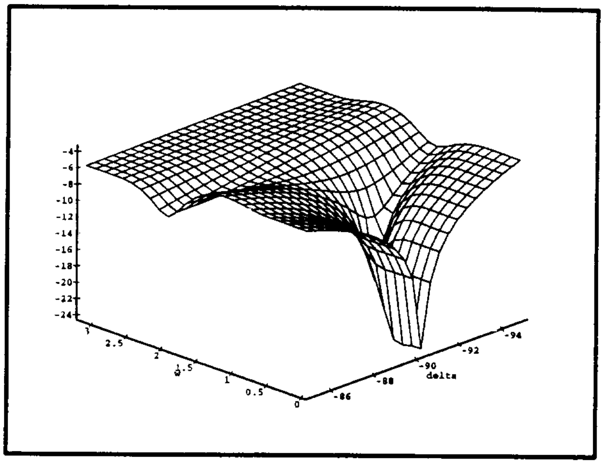

Figure 2: Frequency response of $\widetilde{H}(z)$ for $\{k = 7, -95 \leq \delta/\beta_1 \leq -85\}$

4.1. Causal Implementation

Although equation (8) suggests a non-causal implementation for $\widetilde{H}(z)$, a causal implementation is always feasible. By Lemma (2) (proved in the appendix), we have that $H_n(z)$ can be written in terms of $H_{n-k}, \cdots, H_{n-2k}$. It is then straightforward to show that $\widetilde{H}(z)$ can always be written as a linear scalar combination of

$$\{H_n(z), H_{n-k}(z), \cdots, H_{n-2k}(z)\}.$$

or

$$\widetilde{H}(z) = \beta_1 H_n(z) + \sum_{j=k}^{2k} a_j H_{n-j}(z),$$

where $a_j \in \mathbb{C}$ for all j. In other words, the complementary filter can be achieved by tapping the output of the adders of the lattice structure given in Figure (1) and adding a proper linear combination of these outputs.

In summary, a two-band bi-orthogonal filter bank where both filters are of order n can be uniquely characterized by the following $n+2$ parameter set

$$\{b_1, b_2, \cdots, b_n, \delta/\beta_1 \in \mathbb{C}, k \in \{0, \cdots, n-1\}\}.$$

5. CONCLUSIONS

We have investigated relationships among Padé table, continued fraction expansion and two-channel perfect reconstruction filter banks. Through characterization of all two-channel bi-orthogonal PR filter banks, we developed a new lattice structure for this class of filter banks. The parameter space found using this method is unique for each filter bank. Similarly to any other lattice structure, the PR property is achieved structurally and quantization of the parameters of the lattice does not effect this property. We showed that for a given filter, the set of all complementary filters can be uniquely specified by two parameters, namely the end-to-end delay of the system and a scalar quantity.

6. APPENDIX

In this Appendix, the outline of the proof of Theorem (2) is given. Let us define

$$G_1(z) = \widetilde{H}(z) - \beta_1 H_n(z)$$

where β_1 is chosen such that $g_1(n) = 0$. By direct substitution,

$$H_n(z)G_1(-z) - G_1(z)H_n(-z) = \alpha z^{2k+1}$$

so that $G_1(z)$ and $H_n(z)$ form a PR pair. Now if $k = n-1$, $g_1(n-1)$ cannot be zero, so that by Lemma (1), $G_1(z) = \delta H_{n-1}(z)$ for some $\delta \in \mathbb{C}$, $\widetilde{H}(z) = \delta H_{n-1}(z) + \beta_1 H_n(z)$ satisfying (8) and we are done.
If $k < n-1$, then we define

$$G_i(z) = z^2 G_{i-1}(z) - \beta_i H_n(z), \qquad i = 2, 3, \cdots, n-k \quad (9)$$

where β_i is chosen such that $g_i(n) = 0$. Now,

$$H_n(z)G_i(-z) - G_i(z)H_n(-z) = \alpha z^{2(k+i)-1}$$

which implies that $g_i(n-1) = 0$ for $i < n-k$ and $g_i(n-1) \neq 0$ for $i = n-k$. This also implies that $G_i(z)$ is always at most of order $n-1$. By Lemma (1), $G_{n-k}(z) = \delta H_{n-1}(z)$, and repeated substitution of (9) leads to (8).

We now show that if $\widetilde{H}(z)$ is causal then k, δ and β_1 are the only degrees of freedom. We first show that the following lemma is true.

Lemma 2 *For a given sequence of polynomials $\{H_n(z)\}$ satisfying (7), it is possible to write $H_n(z)$ as*

$$\sum_{j=k}^{2k} a_j z^{2(j-k)} H_{n-j}(z),$$

for all $k \geq 1$, where $a_j \in \mathbb{C}$, $H_0(z) = H_{-1}(z) = 1$ and $H_i(z) = 0$ for $i \leq -2$.

Proof: $k = 1$ corresponds to (7). Now, assuming that the above lemma is true for arbitrary k, by mathematical induction it is straightforward to show that it also holds for $k+1$.

Equipped with Lemma (2), we can now complete the proof of the theorem. By regrouping different terms of (8), we have

$$\begin{aligned}\widetilde{H}(z) &= (\delta H_{n-1}(z) + \beta_{n-k} H_n(z)) z^{-2(n-k-1)} \\ &+ \left(\sum_{j=0}^{n-k-2} \beta_{j+1} z^{-2j}\right) H_n(z), \qquad (10)\end{aligned}$$

and using Lemma (2), it is possible to choose β_{n-k} such that

$$\begin{aligned}\widetilde{H}(z) &= \gamma_1 H_{n-2}(z) z^{-2(n-k-2)} \\ &+ \left(\sum_{j=0}^{n-k-2} \beta_{j+1} z^{-2j}\right) H_n(z). \qquad (11)\end{aligned}$$

By iteratively applying the same procedure, the RHS will only contain positive power of z. It is clear that for this and only this specific choice of $\beta_{n-k}, \beta_{n-k-1}, \cdots, \beta_2$, $\widetilde{H}(z)$ is causal. This completes the proof of Theorem (2).

7. REFERENCES

[1] P. P. Vaidyanathan, "Theory and design of M channel maximally decimated QMF with arbitrary M, having perfect reconstruction property," *IEEE Trans. on ASSP*, No. 4, April 1987.

[2] P. P. Vaidyanathan, "Causal FIR matrices with anticausal FIR inverses, and application in characterization of bi-orthonormal filter banks", *Proceedings of ICASSP*, pp. 177-180, 1994.

[3] M. Vetterli and D. J. LeGall, "Perfect reconstruction FIR filter banks: Some properties and factorizations," *IEEE Trans. on ASSP*, Vol. ASSP-37, No. 7, July 1989.

[4] A.C.M. Kalker and I.Shah, "On ladder structures and linear phase condition for multidimensional biorthogonal Filter Banks", *Submitted to IEEE Trans. on Signal Processing*, 1994.

[5] M. Vetterli and C. Herley, " Wavelets and filter banks: theory and design ", *IEEE Trans. on Signal Processing*, No. 9, pp. 2207-2232, Sept. 1992.

[6] G.A. Baker, Jr., *Essentials of Padé Approximants*, Academic Press, 1975.

[7] L. Lorentzen and H. Waadeland, *Continued Fractions with Applications*, Studies in computational mathematics 3, North-Holland, 1992.

[8] A. Magnus, " Certain continued fractions associated with the Padé table ", *Math. Zeitschr. 78*, 1962, pp. 361-374.

	$H_7(z)$	$H_6(z)$	$H_5(z)$
0	1.0000000000	1.0000000000	1.0000000000
1	3.1029314858	3.1029314858	3.1029314858
2	2.7384614815	2.8819998520	3.1734177538
3	-0.1214690295	0.3239206941	1.2281704828
4	-0.8118612163	-0.3563540087	0.4564133099
5	0.1338739557	0.3101626433	0.3204724809
6	0.1427351497	0.2082479716	
7	-0.0400009711		

Table 1: The coefficients of D4 filter and its first two approximants

b_1	0.1611379440
b_2	2.2342472102
b_3	1.6658767160
b_4	-2.3332622209
b_5	-1.1149202725
b_6	3.0777960077
b_7	-2.2635601304

Table 2: The value of b_i $(1 \leq i \leq 7)$ for D4 filter