MULTI-DIMENSIONAL, PARAUNITARY PRINCIPAL COMPONENT FILTER BANKS

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ABSTRACT

This paper presents a generalization of the one-dimensional principal component filter bank (PCFB) derived in [4] to higher dimensions. Previously, the results in [4] were extended to two-dimensional signals in [5], but the work in [5] was limited to 2D signals and separable resampling operators. The filter bank discussed here results in minimizing the mean squared error when only Q out of P subbands are retained. Furthermore, it is shown that the filter bank maximizes theoretical coding gain (TCG). Simulations are presented demonstrating the potential of the PCFB.

1. INTRODUCTION

Multirate filter banks are a popular tool in a variety of signal and image processing systems. The design of 1D filter banks design has been thoroughly addressed by the research community[6, 7]. Recently, the design of 2D and higher dimensional filter banks has received a great deal of attention. The majority of these design techniques focus on data independent figures of merit such as aliasing cancellation, magnitude distortion, phase distortion, analysis/synthesis filter approximating ideal bandpass filters, etc. In [4], filter banks are designed based on the statistical properties of input signals so that energy is maximally compacted into the first few filter bank channels (subband components). This results in a Principal Component Filter Bank (PCFB) solution. The resultant filter bank minimizes the mean-squared error when only Q out of P subband channels are used in the reconstruction. The formulation in [4] only considers one dimensional signals. This formulation was later extended in [5] to two dimensional signals where the subsampling matrices were restricted to be separable. Furthermore, the formulation in [5] does not allow for generalization beyond two dimensional signals. Here, the 1D principal component filter bank formulation is generalized to MD signals using non-separable resampling matrices. Moreover, it is proven that the resultant PCFB simultaneously maximizes theoretical coding gain (TCG), and hence, are optimal in an information theoretic sense as well.

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2. MD PRINCIPAL COMPONENT FILTER BANKS

2.1. Minimization of Reconstruction Error

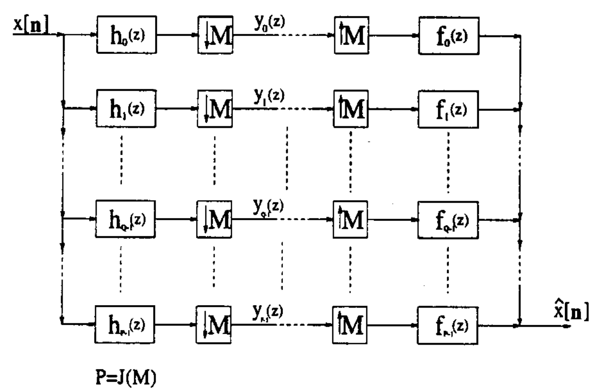

Figure 1: A Typical P-band Paraunitary Filter Bank

Consider the paraunitary [7] filter bank structure shown in Figure 1, where $\mathbf{n}$ is the column vector $[\,n_0 \quad n_1 \quad \cdots \quad n_{d-1}\,]^T$, $\mathbf{z} = [\,z_0 \quad z_1 \quad \cdots \quad z_{d-1}\,]^T$ denotes the d dimensional Z transform variable, M is a $d \times d$ nonsingular integer matrix referred to as a resampling matrix [7], and $J(M)$ denotes the absolute value of the determinant of M. The input signal $x[\mathbf{n}]$ is split into $P = J(M)$ subbands via a bank analysis filters. The outputs of these filters are band limited, and hence can be subsampled at their respective Nyquist rates. The subband signals are then processed. The processed subband signals are upsampled, increasing their sampling rate to the original rate. The upsampled signals are interpolated via the bank of synthesis filters. The outputs of the synthesis filters are averaged to form the reconstructed signal $\hat{x}[\mathbf{n}]$. Given a discrete, MD, random signal, $x[\mathbf{n}]$, and the structure in Figure 1, the goal is to design the analysis and synthesis Q band filters $h_i(\mathbf{z})$, $f_i(\mathbf{z})$, $i = 0, 1, \ldots, Q-1$ that minimize

$$J = \lim_{\mathbf{N}\to\infty} \frac{1}{\prod_0^{d-1} N_i} \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} E\{(\mathbf{x}[\mathbf{n}] - \hat{\mathbf{x}}[\mathbf{n}])^T(\mathbf{x}[\mathbf{n}] - \hat{\mathbf{x}}[\mathbf{n}])\}, \tag{1}$$

where $\mathbf{1}$ is a d dimensional column vector with unity components, and $\mathbf{N} = [N_0 \quad N_1 \quad \cdots \quad N_{d-1}]^T$. The terms $\mathbf{x}[\mathbf{n}]$ and $\hat{\mathbf{x}}[\mathbf{n}]$ are the Type 2 polyphase representations [7] of the input signal $x[\mathbf{n}]$ and the output signal reconstructed from only Q out of P subband components respectively.

Theorem 2.1 *Let $\mathbf{x}[\mathbf{n}]$ be a P-vector, zero mean, second-order process with time varying autocorrelation matrix*

$$R_{\mathbf{XX}}(\mathbf{n};\tau) = E\{\mathbf{x}[\mathbf{n}+\tau]\mathbf{x}^T[\mathbf{n}]\}, \quad (2)$$

such that its entries are uniformly bounded. Let $\bar{R}_{\mathbf{XX}}[\tau]$ be the time averaged autocorrelation matrix

$$\bar{R}_{\mathbf{XX}}(\tau) = \lim_{\mathbf{N}\to\infty} \frac{1}{\prod_0^{d-1} N_i} \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} R_{\mathbf{XX}}(\mathbf{n};\tau) \quad (3)$$

and $\bar{S}_{\mathbf{XX}}(\omega)$ the time averaged spectral density matrix

$$\bar{S}_{\mathbf{XX}}(\omega) = \sum_{\tau=-\infty}^{\infty} \bar{R}_{\mathbf{XX}}(\tau) e^{-j\omega^T\tau}, \quad (4)$$

where, $\omega = [w_0 \quad w_1 \quad \cdots \quad w_{d-1}]^T$. The optimal paraunitary $Q\times P$ and $P \times Q$ matrix filters $H(\omega)$, $F(\omega)$ minimize

$$J = \lim_{\mathbf{N}\to\infty} \frac{1}{\prod_0^{d-1} N_i} \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} E\{(\mathbf{x}[\mathbf{n}] - \hat{\mathbf{x}}[\mathbf{n}])^T(\mathbf{x}[\mathbf{n}] - \hat{\mathbf{x}}[\mathbf{n}])\}, \quad (5)$$

where

$$\hat{\mathbf{x}}[\mathbf{n}] = \sum_{\mathbf{k}} F[\mathbf{n}-\mathbf{k}]\mathbf{y}[\mathbf{k}], \qquad \mathbf{y}[\mathbf{k}] = \sum_{\mathbf{l}} H[\mathbf{k}-\mathbf{l}]\mathbf{x}[\mathbf{l}], \quad (6)$$

if and only if

$$H(\omega) = [\mathbf{v}_0(\omega) \quad \mathbf{v}_1(\omega) \quad \ldots \quad \mathbf{v}_{Q-1}(\omega)]^{T*}\,T(\omega),$$
$$F(\omega) = H^{T*}(\omega), \quad (7)$$

where $\mathbf{v}_i[\omega]$ is the eigenvector corresponding to the $(i+1)$th largest eigenvalue of $\bar{S}_{\mathbf{XX}}(\omega)$ and $T(\omega)$ is any $Q\times Q$ unitary square matrix for all ω.

The filter banks from Theorem 2.1 are called the generalized PCFB's. If $T(\omega) = I$, the filter bank is called the PCFB.
Proof: We can write the cost function (5) in the frequency domain as

$$J = \lim_{\mathbf{N}\to\infty} \frac{1}{\prod_0^{d-1} 2\pi N_i} \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} \int_{\mathbf{0}}^{2\pi\mathbf{1}} tr\{(I - G(\omega))S_{\mathbf{XX}}(\mathbf{n};\omega)(I-G(\omega))^{T*}\}d\omega, \quad (8)$$

where $S_{\mathbf{XX}}(\mathbf{n};\omega)$ is the Fourier transform of $R_{\mathbf{XX}}(\mathbf{n};\tau)$, $G(\omega) = F(\omega)H(\omega)$ is the composite $P\times P$ filter of rank Q, and the integral is for every component of ω. Hence, we need to find the optimal $G(\omega)$ which minimizes (8). From (8),

$$J = \int_{\mathbf{0}}^{2\pi\mathbf{1}} \lim_{\mathbf{N}\to\infty} \frac{1}{\prod_0^{d-1} 2\pi N_i} \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} tr\{(I - G(\omega))S_{\mathbf{XX}}(\mathbf{n};\omega)(I-G(\omega))^{T*}\}d\omega. \quad (9)$$

Finally,

$$J = \frac{1}{(2\pi)^d} \int_{\mathbf{0}}^{2\pi\mathbf{1}} tr\{(I - G(\omega))\bar{S}_{\mathbf{XX}}(\omega)(I-G(\omega))^{T*}\}d\omega, \quad (10)$$

where

$$\bar{S}_{\mathbf{XX}}(\omega) \triangleq \lim_{\mathbf{N}\to\infty} \frac{1}{\prod_0^{d-1} N_i} \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} S_{\mathbf{XX}}(\mathbf{n};\omega). \quad (11)$$

In order to minimize (10), the integrand must be minimized for every ω. The integrand can be written as

$$tr\{(\bar{S}_{\mathbf{XX}}^{1/2}(\omega) - G(\omega)\bar{S}_{\mathbf{XX}}^{1/2}(\omega))(\bar{S}_{\mathbf{XX}}^{1/2}(\omega) - G(\omega)\bar{S}_{\mathbf{XX}}^{1/2}(\omega))^{T*}\}. \quad (12)$$

Then, the rank Q matrix $G(\omega)$ minimizes (12) iff

$$G(\omega) = \sum_{j=0}^{Q-1} \mathbf{v}_j(\omega)\mathbf{v}_j^{T*}(\omega), \quad (13)$$

where $\mathbf{v}_j(\omega)$ is the normalized eigenvector corresponding to the $(j+1)$th largest eigenvalue of $\bar{S}_{\mathbf{XX}}^{1/2}(\omega)$ and hence of $\bar{S}_{\mathbf{XX}}(\omega)$. It is easily verified that $H(\omega)$ and $F(\omega)$ given by (7) are the unique solution of (13). ∎

2.2. Maximization of the Theoretical Coding Gain

In this subsection, we prove the above PCFB maximizes the TCG. First, let us prove the following lemma.

Lemma 2.1 *Given a real-valued signal $x[\mathbf{n}]$ with zero mean and variance σ_x^2, and any invertible transform which maps $x[\mathbf{n}]$ to a series of P signals $\{x_i[\mathbf{n}], i = 0, 1, \ldots, P-1\}$ with variances $\{\sigma_i^2 \neq 0, i = 0, 1, \ldots, P-1\}$, where*

$$\sum_{i=0}^{P-1} \sigma_i^2 = \sigma_x^2.$$

If

$$\forall\ \tilde{\sigma}_i^2, \quad \textit{such that} \quad \sum_{i=0}^{P-1} \tilde{\sigma}_i^2 = \sigma_x^2, \quad (14)$$

and

$$\forall\ 0 \le Q \le P, \quad \textit{such that} \quad \sum_{i=0}^{Q-1} \sigma_i^2 \ge \sum_{i=0}^{Q-1} \tilde{\sigma}_i^2, \quad (15)$$

then

$$\prod_{i=0}^{P-1} \sigma_i^2 \le \prod_{i=0}^{P-1} \tilde{\sigma}_i^2, \quad (16)$$

where the equality holds iff $\tilde{\sigma}_i^2 = \sigma_i^2$, $0 \le i < P$.

Proof: From (15),

$$\sigma_i^2 \ge \sigma_j^2, \quad \text{for} \quad i \le j, \quad 0 \le i,j < P. \quad (17)$$

Without loss of generality, assume

$$\tilde{\sigma}_i^2 \ge \tilde{\sigma}_j^2, \quad \text{for} \quad i \le j, \quad 0 \le i,j < P, \quad (18)$$

and

$$\tilde{\sigma}_i^2 = \sigma_i^2 + \epsilon_i - \epsilon_{i+1}, \qquad \text{for} \qquad i = 0, 1, \ldots, P-1, \qquad (19)$$

with $\epsilon_P = 0$. We have

$$\sum_{i=0}^{P-1} \tilde{\sigma}_i^2 = \sum_{i=0}^{P-1} \sigma_i^2 + \epsilon_0 \Longrightarrow \epsilon_0 = 0, \qquad \text{(due to (14))}$$

and

$$\begin{aligned} \forall\ 1 \le Q \le P, & \\ \sum_{i=0}^{Q-1} \tilde{\sigma}_i^2 &= \sum_{i=0}^{Q-1} \sigma_i^2 - \epsilon_Q \\ &\Longrightarrow \epsilon_Q \ge 0. \quad \text{(from (15))} \qquad (20) \end{aligned}$$

Let

$$f(\epsilon_1, \epsilon_2, \ldots, \epsilon_{P-1}) = \prod_{i=0}^{P-1} \tilde{\sigma}_i^2 = \prod_{i=0}^{P-1} (\sigma_i^2 + \epsilon_i - \epsilon_{i+1}).$$

Then, for $j = 1, 2, \ldots, P-1$,

$$\begin{aligned} \frac{\partial f(\epsilon_1, \epsilon_2, \ldots, \epsilon_{P-1})}{\partial \epsilon_j} &= (\sigma_{j-1}^2 + \epsilon_{j-1} - \epsilon_j) - \\ & \quad (\sigma_j^2 + \epsilon_j - \epsilon_{j+1}) \\ &= \tilde{\sigma}_{j-1}^2 - \tilde{\sigma}_j^2 \ge 0, \end{aligned}$$

The last inequality comes from Inequality (18). Hence, $f(\epsilon_1, \epsilon_2, \ldots, \epsilon_{P-1})$ is nondecreasing with respect to ϵ_j, $j = 1, 2, \ldots, P-1$. Since $\epsilon_j \ge 0$ (Inequality (20)), $j = 1, 2, \ldots, P-1$,

$$\prod_{i=0}^{P-1} \tilde{\sigma}_i^2 \ge \prod_{i=0}^{P-1} \sigma_i^2,$$

where the equality holds iff $\epsilon_i = 0$, $i = 0, 1, \ldots, P$, *i.e.*, $\tilde{\sigma}_i^2 = \sigma_i^2$, $i = 0, 1, \ldots, P-1$. ∎

Now we are ready to prove the PCFB maximizes the TCG.

Theorem 2.2 *The TCG of the MD uniform paraunitary filter bank as in Figure 1 with resampling matrix M for a zero mean cyclo-wide-sense stationary input signal $x[\mathbf{n}]$ with a periodicity matrix M is maximized iff it is the PCFB.*

Proof: Denote by $\mathbf{x}[\mathbf{n}]$ the Type 2 polyphase representation of the input signal $x[\mathbf{n}]$, hence $\mathbf{x}[\mathbf{n}]$ is a zero mean vector WSS (wide-sense stationary) process. From the proof of Theorem 2.1, for any $Q \le P$,

$$\begin{aligned} J &= \int_0^{2\pi 1} tr\{(I - G(\omega)) S_{\mathbf{XX}}(\omega) (I - G(\omega))^{T*}\} d\omega, \\ &= \int_0^{2\pi 1} tr\{S_{\mathbf{XX}}(\omega) (I - G(\omega))^{T*} (I - G(\omega))\} d\omega, \\ &= \int_0^{2\pi 1} tr\{S_{\mathbf{XX}}(\omega) G_{tmp}\} d\omega, \end{aligned}$$

where

$$G_{tmp} \stackrel{\text{def}}{=} I - G^{T*}(\omega) - G(\omega) + G^{T*}(\omega) G(\omega)$$

Since

$$G(\omega) = F(\omega) H(\omega) = H^{T*}(\omega) H(\omega),$$

where $H(\omega)$ is a $Q \times P$ unitary matrix (*i.e.*, $H(\omega) H^{T*}(\omega) = I$), the rows of which are the polyphase 1 representations [7] of the first Q subband analysis filters $h_i(\mathbf{z})$, $i = 0, 1, \ldots, Q-1$, we have

$$G_{tmp} = I - H^{T*}(\omega) H(\omega).$$

Hence

$$\begin{aligned} J &= \int_0^{2\pi 1} tr\{S_{\mathbf{XX}}(\omega)(I - H^{T*}(\omega) H(\omega))\} d\omega, \\ &= \int_0^{2\pi 1} tr\{S_{\mathbf{XX}}(\omega) \tilde{H}^{T*}(\omega) \tilde{H}(\omega)\} d\omega, \\ &= \int_0^{2\pi 1} tr\{\tilde{H}(\omega) S_{\mathbf{XX}}(\omega) \tilde{H}^{T*}(\omega)\} d\omega, \\ &= \sum_{i=Q}^{P-1} \sigma_{y_i}^2, \end{aligned}$$

where the rows of $\tilde{H}(\omega)$ are the polyphase 1 [7] representation of the last $P - Q$ subband analysis filters $h_i(\mathbf{z})$, $i = Q, Q+1, \ldots, P-1$, and $\sum_{i=Q}^{P-1} \sigma_{y_i}^2 \stackrel{\text{def}}{=} 0$ for $Q = P$. If and only if the generalized PCFB's are used,

$$J_{min} = \sum_{i=Q}^{P-1} \sigma_{y_i}^2 = \sum_{i=Q}^{P-1} \sigma_{x_i}^2.$$

i.e., $\sum_{i=Q}^{P-1} \sigma_{y_i}^2$ is minimized, or $\sum_{i=0}^{Q-1} \sigma_{y_i}^2$ is maximized because $\sum_{i=0}^{P-1} \sigma_{y_i}^2$ is constant given $\mathbf{x}[\mathbf{n}]$. Moreover, if and only if the PCFB is used, $\sum_{i=0}^{Q-1} \sigma_{y_i}^2$ are maximized for all $Q = 1, 2, \ldots, P$. Other generalized PCFB's do not work for all Q. Hence, from Lemma 2.1, the geometric mean of $\sigma_{y_i}^2$, $i = 0, 1, \ldots, P-1$ is minimized if the PCFB is used, which proves the theorem. ∎

3. SIMULATIONS

The algorithm to generate the PCFB has been designed [8] and implemented. Some of the simulations for the 256×256 cameraman image for all seven different resampling lattices where $J(M) = 4$ are furnished here. A comparison of the PCFB to traditional parallelogram filter banks is shown in Figure 2 and Table 1 in terms of the TCG and the transform efficiency. In the table, $M = [a \quad b; c \quad d]$ is the 2×2 resampling matrix. Figure 3 shows the results for reconstructing an image from only the first subband signal.

4. CLOSING REMARKS AND FUTURE WORK

In this paper a design technique for the MD Principal Component Filter Bank is presented. The resulting principal component filter bank decomposes the input signal into uncorrelated lower rate principal components. The filter bank is optimal in the mean squared sense when a limited number of subband channels are used to reconstruct the original

Resamp. M	Effi. for PCFB	Effi. for Trad. FB
1, [4 0;0 1]	94.5%	48.1%
2, [4 1;0 1]	97.4%	48.1%
3, [4 2;0 1]	97.5%	48.1%
4, [4 3;0 1]	97.3%	48.1%
5, [2 0;0 2]	97.3%	96.9%
6, [2 1;0 2]	97.5%	96.7%
7, [1 0;0 4]	96.3%	48.8%

Table 1: Transform Efficiency for PCFB and Traditional Parallelogram FB for different resampling matrix when only the first subband is kept

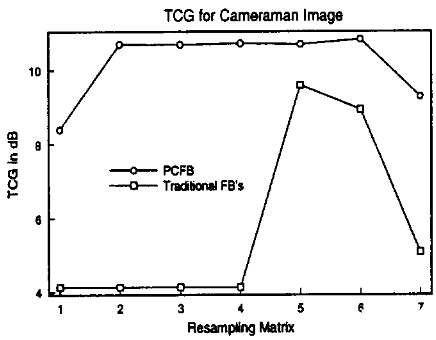

Figure 2: TCG for PCFB and Traditional Parallelogram FB

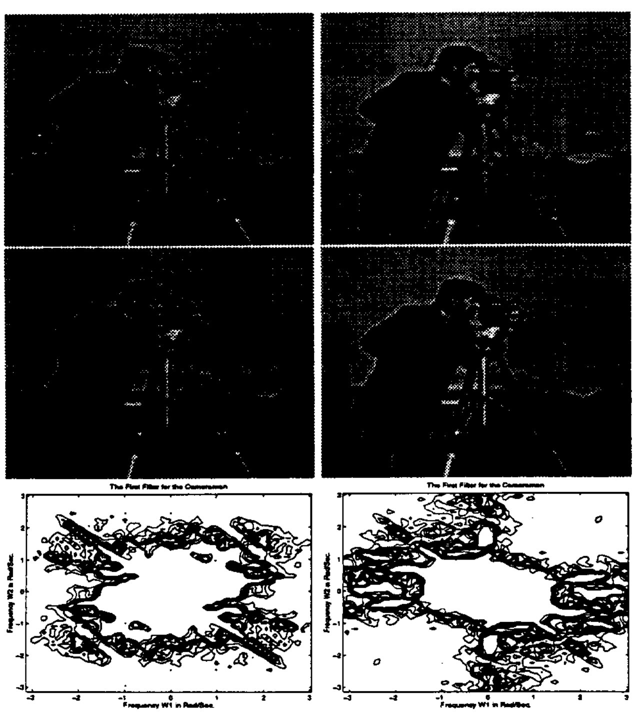

Figure 3: The Reconstructed Image Retaining Only One Subband Using Traditional Parallelogram FB (1st row, left for M5, right for M4) and PCFB (2nd row, left for M5, right for M4) and the First Subband PCFB Filter for M5=[2 0;0 2] (3rd row, left) and M4=[4 3;0 1] (3rd row, right)

signal. It also maximizes the TCG. The simulations indicate that the PCFB depends on the resampling matrix. In [8], we show that the appropriately chosen resampling matrices generate better reconstructed signals. For stationary signals, a theoretical proof [8] shows that the input (spectrum) hardly affects the PCFB.

The MD PCFB can be potentially applied to a wide range of areas, including data compression, data storage, model reduction, feature extraction, pattern classification and segmentation. In addition the design of FIR PCFB's, nonuniform PCFB's, and the efficient parameterization of PCFB parameters is under investigation.

5. ACKNOWLEDGEMENTS

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6. REFERENCES

[1] D. R. Brillinger, *Time Series: Data Analysis and Theory*, Holden-day Inc., San Francisco, 1981.

[2] N. S. Jayant and P. Noll, *Digital Coding of Waveforms: Principles and Applications to Speech and Video*. Englewood Cliffs, NJ: Prentice-Hall Signal Processing Series. 1984.

[3] A. K. Soman and P. P. Vaidyanathan, "Coding Gain in Paraunitary Anlysis/Synthesis Sytems", *IEEE Trans. on Signal Processing*, vol. 41, no. 5, pp. 1824-1835, May 1993.

[4] M. Tsatsanis, *On Wavelets and Time-varying System Identification*, Univ. of Virginia, Ph.D. Thesis, Sept. 1992.

[5] A. Tirakis, A. Delopoulos and S. Kollias, "2-D Filter Bank Design for Optimal Reconstruction Using Limited Subband Information", to appear *IEEE Trans. on Image Processing.*

[6] Mark J. T. Smith and Thomas P. Barnwell, III, "Exact Reconstruction Techniques for Tree-structured Subband Coders", *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. 34, no. 3, pp. 434-441, June 1986.

[7] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, Englewood Cliffs, N. J., 1993.

[8] Bo Xuan and Roberto H. Bamberger, "Multidimensional, Paraunitary Principle Component Filter Banks", to be submitted to *IEEE Trans. on Signal Processing.*