A MEASURE OF NEAR-ORTHOGONALITY OF PR BIORTHOGONAL FILTER BANKS

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ABSTRACT

We study the non-orthogonality of perfect-reconstruction (PR) biorthogonal filter banks by measuring the energy preservation between the spatial and transform domains. The mathematical formulation of that issue leads to the computation of the Riesz constants, and a more relevant modelization leads to a measure of near-orthogonality which is well suited for image compression systems based on filter banks. This provides a criterion for the validity of the energy preservation approximation: we can compare the latter approximation with the one that is made when estimating the perceptual quality of an image by the mean square error.

1. INTRODUCTION

While most of the transforms used in signal processing are orthogonal, some biorthogonal transforms have been introduced in the last few years, among which the most famous are the biorthogonal wavelet transforms. Orthogonal transforms have some nice properties, such as energy preservation, that are often used in quantization procedures and bit allocation algorithms.

These properties make the orthogonal transforms very attractive, but in the case of dyadic wavelets, orthogonality is non-compatible with phase-linearity, which seems to be relevant too. For that reason some authors have tried to synthesize biorthogonal filter banks as orthogonal as possible. As both analysis and synthesis low-pass filters are identical in the orthogonal case, they have tried to make both low-pass filters as similar as possible: less dissimilar length, filter coefficients close to each other [1], $\mathcal{L}^2$-norm [2], qualitative closeness of the frequency responses, with equality at given points [3].

The similarity of both low-pass filters is one out of many properties of orthogonal filter banks, which is not very relevant in itself. Therefore, in the debate about the relevant criteria, we need a relevant measure of the near-orthogonality of biorthogonal filter banks.

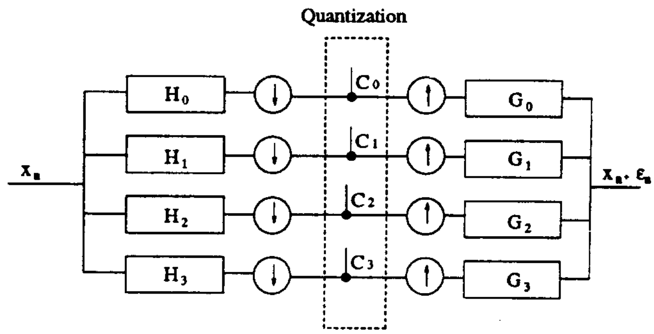

Figure 1: Example of a 4-band filter bank.

Kronander [4] has already proposed an "orthogonality measure", which however does not provide any energy preservation control. Therefore we study near-orthogonality with relation to energy preservation and present three main results:

- The computation of the Riesz constants of the underlying Riesz basis, as a result of the mathematical formulation;
- The definition of a new measure of near-orthogonality, which is well suited for image compression;
- A discussion on the validity of the energy preservation approximation.

2. FILTER BANKS AND RIESZ BASES

We consider M-band filter banks like the one in Figure 1. The analysis part consists of filtering and decimating the signal in each subband. The quantization part is modelled by the addition of a quantization noise, whatever the algorithm might be. The synthesis part consists in upsampling and filtering the signal in each subband and in adding all the subbands. We consider perfect reconstruction (PR) filter banks: Without any quantization error, the reconstructed signal is the original signal, up to a delay. Such PR filter banks may be orthogonal or not. In the latter case, we call them biorthogonal filter banks. We use 1-D notations, but the N-D extension is straightforward.

Energy preservation is the most widely used property of orthogonal transforms for signal compression applications. With a biorthogonal filter bank, this property does not hold anymore, but we want to keep it as an approximation. E.g., let us consider an usual bit allocation algorithm: The bit allocation satisfies rate-distortion optimality. For such an algorithm we assume rate and distortion to be additive over the subbands [5, 6]. In other words, we want the sum of the subband square errors to be close to the reconstruction square error. That means that there exist two constants A and B, close to 1, so that, whatever the quantization might be:

$$A\sum_{j=1}^{M}\sum_{n} c_{j,n}^2 \leq \sum_{n} \varepsilon_n^2 \leq B\sum_{j=1}^{M}\sum_{n} c_{j,n}^2, \qquad (1)$$

where ε denotes the reconstruction error (in the time domain) and c_j the quantization error in the subband j.

In mathematical terms, we consider the filter bank transform as a decomposition over a non-orthogonal discrete basis of the signal space (See [7, chapter 11] for details). Relation (1) means that this basis has to be a Riesz basis: we want to calculate the optimal Riesz constants A and B. Such decompositions have already been studied as Riesz basis [1, 8], but the computation of the optimal constants A and B has not yet been addressed.

Actually, we are able to calculate them by considering the reconstruction error in the frequency domain (cf. Fig. 1):

$$E(\omega) = \sum_{j=1}^{M} C_j(M\omega)G_j(\omega). \qquad (2)$$

The corresponding energy is given by:

$$\int_0^{\pi} |E(\omega)|^2\, d\omega = \int_0^{\pi} \langle C(\omega), S(\omega)C(\omega)\rangle\, d\omega \qquad (3)$$

where we introduced the operator

$$S(\omega) = (S_{ij}(\omega))_{\substack{1\leq i\leq M\\ 1\leq j\leq M}} \qquad (4)$$

$$S_{ij}(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} G_i\left(\frac{\omega+2k\pi}{M}\right) G_j^*\left(\frac{\omega+2k\pi}{M}\right) \qquad (5)$$

That means that the optimal constant B is the supremum of the spectral radius of the operator $M(\omega)$. We denote the spectrum of $M(\omega)$ by $\mathrm{Sp}(M(\omega))$ and we obtain:

$$B = \sup_{\omega\in[0,\pi]}\left[\max_{\lambda\in \mathrm{Sp}(M(\omega))} |\lambda|\right] \qquad (6)$$

$$A = \inf_{\omega\in[0,\pi]}\left[\min_{\lambda\in \mathrm{Sp}(M(\omega))} |\lambda|\right] \qquad (7)$$

This result can be extended to the multidimensional case. We can remark that in the usual two-band case the eigenvalues of $M(\omega)$ can be computed explicitly, and that: $AB = 1$.

Application to iterated filter banks The major motivation for this study is the use of biorthogonal wavelet transforms. In order to apply our results to iterated filter banks, we consider the equivalent parallel filter banks. We need filter banks with the same decimation rate in all subbands. Therefore we apply our results to the 2^J-band filter bank made of the filters:

$$H_1(\omega) = \prod_{l=0}^{J-1} G_0\left(2^l\omega\right) \qquad (8)$$

$$H_{2^j+k+1}(\omega) = e^{2^{J-j}k\omega} G_1\left(2^{J-j-1}\omega\right) \prod_{l=0}^{J-j-2} G_0\left(2^l\omega\right)$$
$$k = 0\ldots 2^j-1,\; j = 0\ldots J-1$$

which can be shown to hold the same Riesz constants as the iterated filter bank.

There is no simple link between the Riesz constants B_J of the iterated filter bank and B_1 of the simple 2-band filter bank, except that $B_J \leq B_1^J$, which is straightforward by induction. A complementary result can be obtained for regular wavelet filter banks: We prove in [9] that B_J increases to a finite limit as J increases to $(+\infty)$.

As a conclusion we obtained in that section a measure of near-orthogonality which is easy to calculate and based on a precise mathematical background. This measure holds for each application implying the energy preservation principle and for each biorthogonal PR filter bank transform.

3. QUANTIZATION NOISE MODELIZATION AND MEASURE OF NEAR-ORTHOGONALITY

In this section, we study to what extent the Riesz constants are an adequate measure of near-orthogonality for signal compression applications and we derive a more relevant one. The mathematical measure of near-orthogonality is based on the worst situation, corresponding to very particular quantization noises. It is not very relevant to compare such quantization noises, in order to measure the approximation which is done when taking the sum of the subband distortions as the global distortion in the bit allocation algorithm: We would rather use an average value, based on a mathematical expectation.

We firstly assume that the quantization is done independently in each subband, so that the interference terms have a zero mean, and secondly that in each subband the energy is distributed on all the frequency spectrum. However, the energy N_j may be different from one subband to another. It is quite common in image compression to quantify differently different subbands, according to the statistics of the signal and to some psychophysical properties.

With that new modelization of the quantization noise, we obtain as the mathematical expectation of the error:

$$\int_0^{\pi} |E(\omega)|^2 d\omega = \sum_{j=1}^{M} \frac{N_j}{M} \int_0^{\pi} \sum_{k=0}^{M-1} \left| G_j \left(\frac{\omega + 2k\pi}{M} \right) \right|^2 d\omega \quad (9)$$

We control therefore the resulting error as:

$$\sum_n \varepsilon_n^2 \leq NOM \sum_{j=1}^{M} N_j, \quad (10)$$

where we defined the near-orthogonality measure by

$$NOM = \max_{j=1 \ldots M} \sum_n |g_n^j|^2 \quad (11)$$

If we suppose that the noise has the same energy in all the subbands, that means that the quantization noise is completely white, and from another point of view, it is easy to prove, using the perfect-reconstruction property, that the resulting formula is equivalent to the measure of the square error between both low-pass filters. This is the criterion of near-orthogonality that has been used in some former syntheses [2]. As we want to take into account possible differences of energy between the subbands, we measure the near-orthogonality with NOM.

Some other measures might be derived from our method with different modelizations of the quantization noise, but this one is very simple and corresponds to a reasonable noise modelization.

Extension to iterated filter banks

The extension to iterated filter banks is straightforward and consists of considering the equivalent parallel filter banks, made of the synthesis filters G_j, and in defining NOM_J as:

$$NOM_J = \max_{j=1 \ldots J+1} \sum_n |g_n^j|^2 \quad (12)$$

Extension to 2-dimensional filter banks

The results of that section are easy to extend to 2-dimensional filter banks:

$$NOM = \max_{j=1 \ldots M} \sum_{n,m} |g_{n,m}^j|^2 \quad (13)$$

The NOM of a separable filter bank made from a 1-dimensional filter bank with $NOM = m_1$, is m_1^2.

4. VALIDITY OF THE ENERGY PRESERVATION APPROXIMATION

The aim of our study of near-orthogonality is to see when the approximation of energy preservation is admissible: We look for the NOM values that let us assume the energy preservation with an accuracy which is significant for image compression. This significant accuracy of the SNR of the reconstructed image is an open question, but one usually does not see any quality difference between reconstructed images when the SNRs differ with less than 0.5 dB (1 dB for high quality reconstruction). That means that, when allocating the bits, the estimation of the resulting distortion may be 0.5 dB inaccurate. The accuracy of this estimation is given by the near-orthogonality measure, so that the quantization energy preservation is a valid approximation if $NOM \leq 1.12$ (1.26 for high quality reconstruction).

We tested the measure on different 4-band separable PR filter banks, with the usual dyadic iteration scheme. The orthogonality measure only depends on the synthesis filters, so that the results are generally different when you exchange analysis and synthesis filters. However, in the dyadic case without iteration, there is no difference because of the relationship between analysis and synthesis filters [1]: The low-pass analysis filter has the same energy as the high-pass synthesis filter. The results are presented in Table 1. We see that some iterated wavelet transforms may be considered as nearly-orthogonal, even for low-rate compression ($NOM \leq 1.12$), and some others only for high quality compression ($NOM \leq 1.26$).

5. CONCLUSION

We addressed in this paper the issue of measuring how orthogonal a biorthogonal transform is: The measure

	Riesz A	Riesz B	Near-Orthogonality Measure NOM No iteration	4 it. over H_0	4 it. over G_0
Cohen-Daubechies 7-9 [1]	0.57	1.75	1.08	1.12	1.20
Burt-Daubechies-Cohen 5-7 [1]	0.91	1.10	1.04	1.04	1.09
Vetterli 20-24 [10]	0.13	7.83	2.40	2.40	5.14
Vetterli 18-18 [10]	0.33	3.00	1.43	1.88	3.85
Nguyen 23-25 [11]	0.24	4.23	1.27	1.27	1.52
Ikehara [12]	0.60	1.69	1.09	1.09	1.21
Le Bihan-Rioul [2] (Similarity)	0.50	1.99	1.09	1.09	1.13
Le Bihan-Rioul [2] (Frequency Selectivity)	0.28	3.62	1.22	1.22	1.28
Onno (Frequency Selectivity) [13]	0.28	3.58	1.11	1.21	1.26
Onno (Regularity) [13]	0.30	3.32	1.11	1.19	1.23

Table 1: Near-orthogonality measures of 2-D separable PR filter banks constructed from a set of "classical" filter banks.

may depend on the application and on the modelization, and we propose a simple one, well suited for image compression. It is derived through a modelization of the quantization noise from the principle of the Riesz constants computation.

We get from that measure a criterion for the validity of the quantization energy preservation. As the reconstruction quality decreases, the condition gets stronger. Especially for low-rate image compression, one need to take the measure of near-orthogonality into account in the choice of the transform, if the energy preservation property is used in the quantization algorithm

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