MODULATED 2 DIMENSIONAL PERFECT RECONSTRUCTION FIR FILTER BANKS WITH PERMISSIBLE PASSBANDS

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ABSTRACT

In this paper, I consider about the theory of modulated 2 Dimensional (2-D) perfect reconstruction (PR) filters banks with permissible passbands. At first, I design a 2-D complex digital filter with half passband obtained by the sampling matrix. Next, 2-D analysis filter banks are realized by modulating this prototype 2-D complex digital filter and by taking the real part of the output. It is also shown that the modulation in 2-D frequency plane is equivalent to 1-D DFT. A necessary and sufficient condition for 2-D perfect reconstruction filter banks is derived. Finnaly I show some examples.

1. INTRODUCTION

Analysis/synthesis multirate filter banks find applications in a wide variety of digital signal processing systems. 1-D filter banks have been well studied and various design approaches have been successfully developed. Recently, the concept of 1D filter bank is extended to the multidimensional case including 2-D. In particular, I proposed a design method of 2-D PR filter banks using 2-D N-th band digital filter and 2-D DFT. However time -reversal operators have been used and the analyzed signal is complex because of 2-D DFT[1]. On the other hand, the method mentioned in [2] is very efficient but, with finite impulse response filters, the resulting filter banks can only achieve approximate reconstruction. Also I proposed cosine-modulated 2-D PR filter banks for processing real signal[3]. Although this method is very efficient, it is said that the passband supports are nonpermissible[4]. That is, there are some spikes in the stopband.

In this paper, I present a design method of 2-D PR filter banks with permissible passbands by modulating a prototype complex 2-D FIR digital filter. It is very difficult to design 2-D PR filter banks directly because the number of coefficients is more than 1-D PR filter banks. Then I design a complex 2-D digital filter as prototype which have triangular passband. It is shown that 2-D PR filter banks can be realized by modulating a complex prototype filter and by taking the real part of the output. The modulation matrix in 2-D frequency plane is equivalent to 1-D DFT. Finally, a necessary and sufficient condition for 2-D PR filter banks is derived. I show some examples and apply to the subband coding.

2. PRELIMINARY

I simply explain 2-D sampling theory[6][7]. In 2-D plane, the integer lattice Λ is defined to be the set of all integer vector $\mathbf{n} = (n_1, n_2)^T$ With $\mathbf{z}$ defined as vector, $\mathbf{z} = (z_1, z_2)^T$ and $\mathbf{z}^{\mathbf{n}}$ defined by $\mathbf{z}^{\mathbf{n}} = z_1^{n_1} z_2^{n_2}$. Assuming the sampling matrix be as follows,

$$\mathbf{D} = [\mathbf{d}_0 \ \mathbf{d}_1] = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad (1)$$

z^{D} is defined by

$$\mathbf{z}^{\mathbf{D}} = [z_1^{a} z_2^{b}, z_1^{c} z_2^{d}]^T \quad (2)$$

Fig.1 shows 2-D analysis/synthesis filter bank. When $x(\mathbf{n}) = y(\mathbf{n})$, it is said that the system achieves perfect reconstruction. The Furrier transforms of y(n), denoted as $Y(\omega)$, can be expressed in terms of the Furrier transforms of the input signal and filters as

$$Y(\omega) = \frac{1}{D} \sum_{j=0}^{D-1} \sum_{i=0}^{D-1} X(\boldsymbol{\omega} - \boldsymbol{\omega}_i) H_j(\boldsymbol{\omega} - \boldsymbol{\omega}_i) F_j(\boldsymbol{\omega}) \quad (3),$$

where $D = |\det(\mathbf{D})|$, $\boldsymbol{\omega} = (\omega_1, \omega_2)^T$ and $\boldsymbol{\omega}_i$ are the aliasing offsets, $\boldsymbol{\omega}_i = 2\pi \mathbf{D}^{-T} \mathbf{k}_i$, The vectors $\mathbf{k}_i$ belong to $\aleph(\mathbf{D}^T)$, which is the set of all integer vectors of the form

$$\mathbf{D}^T \mathbf{x} \ \text{ for } \mathbf{x} \in [0,1)^2 . \quad (4)$$

Fig. 2 shows some decomposition schemes for decimation by the sampling matrix D=[2 1;-1 1] with analysis filters having a real impulse response. $H_0(\omega)$ (including origin) typically has passband in the region.

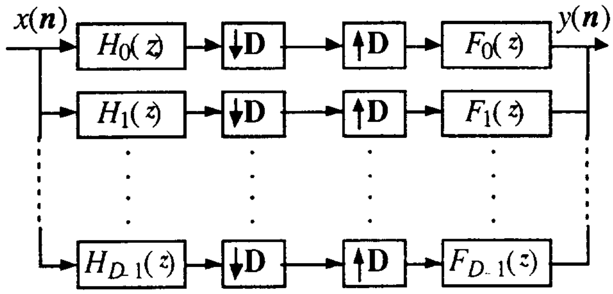

Fig.1 2-D Maximaly decimated filter bank

$$\omega = \pi \mathbf{D}^{-T}\mathbf{x} \quad \mathbf{x} \in [-1\ \ 1)^T \tag{5}$$

Boldfaced letters denote matrices and column vector, with upper and lower case, respectively. The notation $\tilde{\mathbf{H}}(\mathbf{z})$ stands for $\mathbf{H}_*^T(\mathbf{z}^{-1})$, where T and * denote transposition and conjugation .

3 MODULATED 2-D FILTER BANKS

A . Decomposition and Modulation

Fig.2 (a) shows the decomposition scheme whose passbands are symmetric about $\omega = 0$. The filter bank with this decomposition scheme can be easily implemented by cosine-modulating the prototype filter of Fig3(a)[3]. But It is said that this filter bank has nonpermissible passbands[4]. That is, the mgnitude response of these filter banks have some spikes and can not be good. On the other hand, it is known in [4] that the passband in Fig.2(b),(c) has permissible supports.

In this paper, we consider the decomposition schemes with permissible passband supports as shown in Fig. 2(b),(c). To implement this decomposition, I use 2-D complex digital filters with the triangle passband of Fig.3 (b) and (c) as prototype filter, respectively. Passbands of Fig2.(b) and (c) are obtained by modulating the prototype filters of Fig 3.(b),(c) and by taking the real part of the output of the modulated prototype filters. Fig.4 shows simple example with the sampling matrix D=[2 1:-1 1]. So I call these decomposition schemes of Fig.2 (b) and (c) as Type 1 and Type 2, respectively.

B. Design of Complex Prototype Filter

The complex transfer functions of the prototype filters with the passband of Fig3 (b),(c) are expressed by the polyphase components

$$H(\mathbf{z}) = \sum_{j=0}^{D-1} \mathbf{z}^{-\mathbf{n}_j}\left\{F_j(\mathbf{z}^{\mathbf{D}}) + i\,G_j(\mathbf{z}^{\mathbf{D}})\right\} \tag{6}$$

where $F_j(z)$ and $G_j(z)$ are the real and imaginary part of

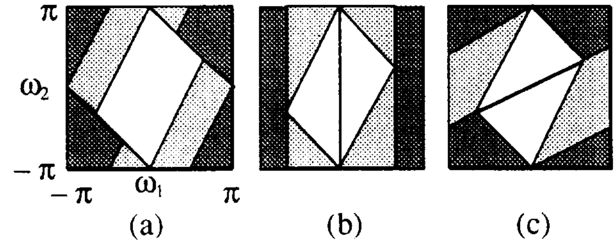

Fig.2 The decomposition schemes of 2-D filter banks

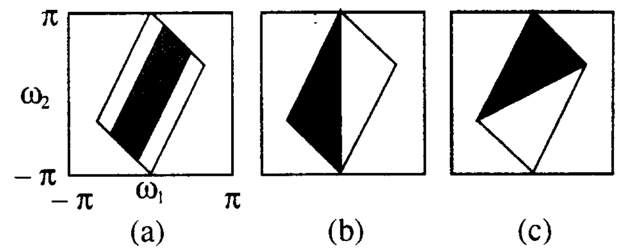

Fig.3 The prototype filyters

the prototype filter, respectively. $\mathbf{n}_j$ are delay vectors expressed by $\mathbf{n}_j \in \aleph(\mathbf{E})$. But delay vectors are not unique as described later.

At first, we compose the complex analysis filters by modulating the prototype filter of Fig.3(c) as shown in Fig.4. Then each shift component of the complex analysis bank is defined by

$$2\pi\mathbf{D}^{-T}\mathbf{k}_l \tag{7}$$

Sifting by the above component corresponds to the following transformation,

$$\mathbf{z} \rightarrow \exp\left[j\,2\pi\mathbf{D}^{-T}\mathbf{k}_l\right] \otimes \mathbf{z} \tag{8}$$

where $[a,b]^T \otimes z = [az_1, bz_2]^T$. Substituting this transformation into (6), each transfer function of the analysis filters can be written by

$$\hat{H}_l(\mathbf{z}) = \sum_{j=0}^{D-1} \exp\left[i\,2\pi\mathbf{n}_j^T\mathbf{D}^{-T}\mathbf{k}_l\right]\mathbf{z}^{-\mathbf{n}_j}\left\{F_j(\mathbf{z}^{\mathbf{D}}) + iG_j(\mathbf{z}^{\mathbf{D}})\right\} \tag{9}$$

Then I show an example. Let the sampling matrix be D=[2 1;-1 1], 3 channel filter bank is obtained. Then the shift vectors $\mathbf{k}_l$ and delay vectors $\mathbf{n}_j$ are

$$\mathbf{k}_l = \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \qquad \mathbf{n}_j = \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}$$

and shift components are

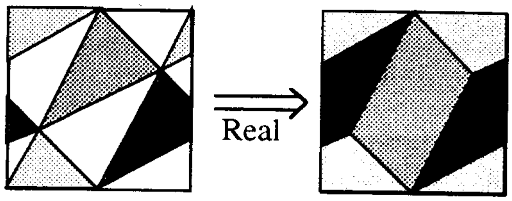

Fig.4 Permisible passband

$$\left[2\pi \mathbf{n}_j^T \mathbf{D}^{-T}\mathbf{k}_l\right]_{l,j} = \frac{2\pi}{3}\begin{bmatrix}0\\1\\2\end{bmatrix}[0\ 1\ 2]$$

It is surprised that this modulated matrix is same as 1D discrete Furrier transform. Therefore 2-D complex analysis filter banks can be expressed as follows..

$$\hat{H}_l(\mathbf{z}) = \sum_{j=0}^{D-1} \exp\left\{ i\,\frac{2l\pi}{D}\left(j-\frac{D-1}{2}\right)+\frac{\pi}{4}\right\} \mathbf{z}^{\mathbf{n}_j}\left\{F_j(\mathbf{z}^{\mathbf{D}}) + iG_j(\mathbf{z}^{\mathbf{D}})\right\} \tag{10}$$

where $\pi/4$ is the phase constraint to ensure that the alias and the phase distortion can be canceled. Finally, the analysis filter banks can be implemented by taking the real part of the complex analysis banks as follows

$$H_l(z) = \mathrm{Re}\,\{\hat{H}_l(z)\}$$

Then the analysis filter banks can be expressed in matrix form as

$$\mathbf{h}(\mathbf{z}) = \left[\mathbf{C}\,\mathbf{f}(\mathbf{z}^{\mathbf{D}}) + \mathbf{S}\mathbf{g}(\mathbf{z}^{\mathbf{D}})\right]\mathbf{e}(\mathbf{z}) \tag{11}$$

where, $\mathbf{e}(\mathbf{z}) = [1\,,\mathbf{z}^{-\mathbf{n}_1}, \cdots \mathbf{z}^{-\mathbf{n}_{D-1}}]^T$

$$\mathbf{h}(\mathbf{z}) = [H_0(\mathbf{z})\,, H_1(\mathbf{z}), \cdots, H_{D-1}(\mathbf{z})]^T$$

$$\mathbf{f}(\mathbf{z}) = \mathrm{diag}\,[F_0(\mathbf{z})\,, F_1(\mathbf{z}), \cdots, F_{D-1}(\mathbf{z})]$$

$$\mathbf{g}(\mathbf{z}) = \mathrm{diag}\,[G_0(\mathbf{z})\,, G_1(\mathbf{z}), \cdots, G_{D-1}(\mathbf{z})]$$

$$\left[\mathbf{C}\right]_{l,j} = \cos\left\{\frac{2l\pi}{D}\left(j-\frac{D-1}{2}\right)+\frac{\pi}{4}\right\}$$

$$\left[\mathbf{S}\right]_{l,j} = \sin\left\{\frac{2l\pi}{D}\left(j-\frac{D-1}{2}\right)+\frac{\pi}{4}\right\}$$

4 . THE CONDITION FOR PERFECT RECONSTRUCTION

I showed that the modulated matrix is equivalent to 1D DFT. To achieve perfect reconstruction, I impose the linear phase condition on the prototype filter $H(\mathbf{z})$. So the following two conditions are required so that $H(\mathbf{z})$ has linear phase.

1) $\mathbf{n}_i + \mathbf{n}_{D-1-i} = 2\mathbf{n}_s$

2) $F_i(\mathbf{z}) = z_1^{-N} z_2^{-M} \tilde{F}_{D-1-i}(\mathbf{z})$,

$G_i(\mathbf{z}) = -\, z_1^{-N} z_2^{-M} \tilde{G}_{D-1-i}(\mathbf{z})\,, G_{(D-1)/2}(\mathbf{z}) = 0$ for D:odd

where $\mathbf{n}_s$ is real vector and (N, M) are the order of $F(\mathbf{z})$ and $G(\mathbf{z})$. The condition 1) is not always satisfied on all delay vectors defined by $\aleph(\boldsymbol{E})$. In this case we must decide the delay vectors so that the condition 1) is satisfied. Although this condition dose not necessarily need as shown later, it is important to design easily the response of Fig.3(b)(c). The condition 2) shows that $F_i(\mathbf{z})$ is time-reversed version of $F_{D-1-i}(\mathbf{z})$. This is important to cancel the alias.

From (11), the polyphase component matrix $\mathbf{E}(\mathbf{z})$ of the analysis filter bank is expressed by

$$\mathbf{E}(\mathbf{z}) = \mathbf{C}\,\mathbf{f}(\mathbf{z}) + \mathbf{S}\mathbf{g}(\mathbf{z}) \tag{12}$$

Then the modulation matrixes have the following relationship

$$\mathbf{C}^T\mathbf{C} = \mathbf{S}^T\mathbf{S} = \mathbf{I}\,,\ \mathbf{C}^T\mathbf{S} = \mathbf{S}^T\mathbf{C} = \mathbf{J}$$

where $\mathbf{J}_D$ is the inverse diagonal matrix.

It is known from [5] that if $\mathbf{E}(\mathbf{z})$, the polyphase component matrix of the analysis bank is lossless,i.e. $\tilde{\mathbf{E}}(\mathbf{z})\mathbf{E}(\mathbf{z}) = \mathbf{I}_\mathbf{D}$, then 2-D PR filter banks can be obtained. Then I get

$$\tilde{\mathbf{E}}(\mathbf{z})\mathbf{E}(\mathbf{z}) = \left\{\tilde{\mathbf{f}}(\mathbf{z})\mathbf{C}^T\mathbf{C}\mathbf{f}(\mathbf{z}) + \tilde{\mathbf{g}}(\mathbf{z})\mathbf{S}^T\mathbf{S}\mathbf{g}(\mathbf{z})\right\} + \left\{\tilde{\mathbf{f}}(\mathbf{z})\mathbf{C}^T\mathbf{S}\mathbf{g}(\mathbf{z}) + \tilde{\mathbf{g}}(\mathbf{z})\mathbf{S}^T\mathbf{C}\mathbf{f}(\mathbf{z})\right\} \tag{13}$$

and from condition 2), the following relation is kept

$$\tilde{\mathbf{f}}(\mathbf{z}) = z_1^N z_2^M \mathbf{J}\mathbf{f}(\mathbf{z})\mathbf{J} \qquad \tilde{\mathbf{g}}(\mathbf{z}) = -z_1^N z_2^M \mathbf{J}\mathbf{g}(\mathbf{z})\mathbf{J}$$

Therefore, the second term on the right side of (13) is

$$\tilde{\mathbf{f}}(\mathbf{z})\mathbf{C}^T\mathbf{S}\mathbf{g}(\mathbf{z}) + \tilde{\mathbf{g}}(\mathbf{z})\mathbf{S}^T\mathbf{C}\mathbf{f}(\mathbf{z}) = z_1^N z_2^M \mathbf{J}\left\{\mathbf{f}(\mathbf{z})\mathbf{g}(\mathbf{z}) - \mathbf{g}(\mathbf{z})\mathbf{f}(\mathbf{z})\right\} = \mathbf{0} \tag{14}$$

As result,

$$\tilde{\mathbf{E}}(\mathbf{z})\mathbf{E}(\mathbf{z}) = \tilde{\mathbf{f}}(\mathbf{z})\,\mathbf{f}(\mathbf{z}) + \tilde{\mathbf{g}}(\mathbf{z})\,\mathbf{g}(\mathbf{z})$$

I get a necessary and sufficient condition for 2-D modulated PR filter banks as follows

$$\tilde{F}_j(\mathbf{z})F_j(\mathbf{z}) + \tilde{G}_j(\mathbf{z})G_j(\mathbf{z}) = 1 \qquad 0 \le j \le D-1 \tag{17}$$

However, owing to the linear phase symmetry of $H(\mathbf{z})$, approximately half of the D constraints are redundant. Therefore, removing the redundant constraints, (17) can be expressed as

$$\tilde{F}_j(\mathbf{z})F_j(\mathbf{z}) + \tilde{G}_j(\mathbf{z})G_j(\mathbf{z}) = 1 \qquad 0 \le j \le \left\lfloor\frac{D-1}{2}\right\rfloor \tag{18}$$

$$\tilde{F}_{(D-1)/2}(\mathbf{z})F_{(D-1)/2}(\mathbf{z}) = 1 \quad \textit{for } D\text{:odd}$$

In this way, we can design 2-D PR filter banks with permissible passbands if the real and imaginary polyphase filter pairs of the complex prototype filter have doubly-complement.

4. DESIGN EXAMPLE

We now present a design example with the sampling matrix D=[2 2; -1 1] which defines the so-called hexagonal decimation. Let N=M=4, we really obtained the permissible supports for this hexagonal decimation and their desired frquency response are shown in Fig.5(a). We design the 2-D filter bank by using the constrained least-square minimization method of matlab. Fig.5(b) shows the obtained magnitude response. It can be seen clearly from this figure that the desired filters are being approximate.

5. CONCLUSION

In this paper I present a new method of modulated 2-D perfect reconstruction filter banks with permissible passbands. At first, I design a 2-D complex digital filter with half passband obtained by the sampling matrix. Next, ad analysis filter banks are realized by modulating this complex prototype 2-D digital filter and by taking real part of the output. It is shown that the modulation in ad frequency plane is equivalent to 1-D DFT. 2-D PR filter banks can be realized by designing only prototype filter whose real and imaginary polyphase filter pairs have doubly-complement.

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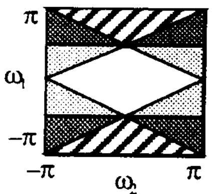

Fig.5(a) The desired decomposition sheme with the sampling matrix D=[2 2; -1 1]

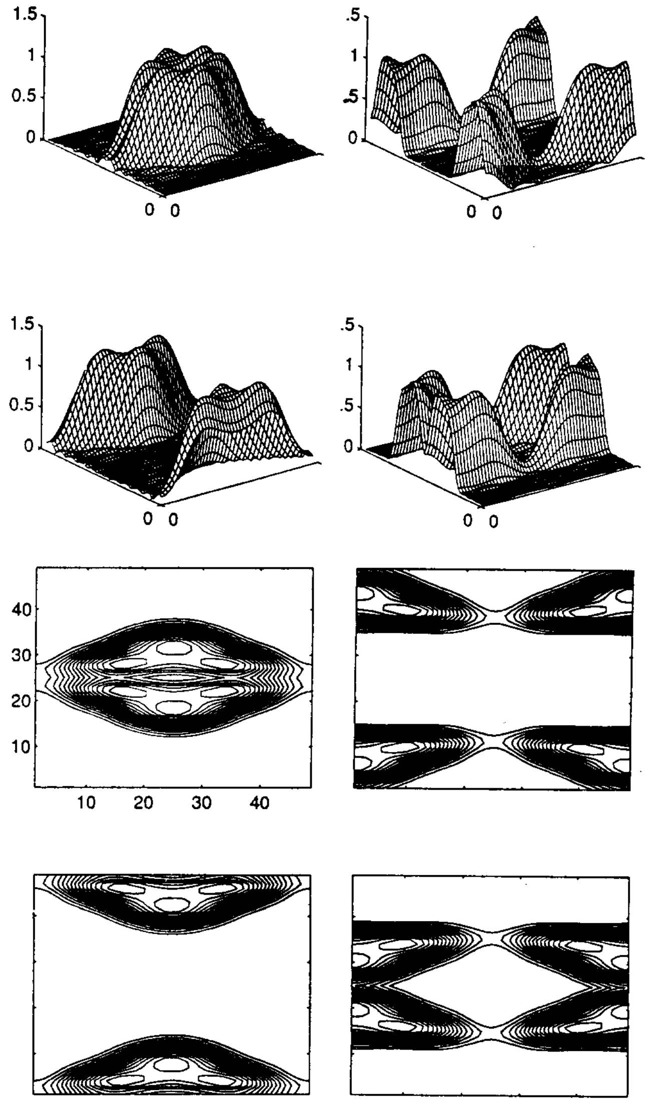

Fig5(b) The designed magnitude response