REGULAR MULTIDIMENSIONAL LINEAR PHASE FIR DIGITAL FILTER BANKS AND WAVELET BASES

T. Cooklev, A. Nishihara
Dept. Physical Electronics
Tokyo Inst. Technology
2-12-1 Ookayama, Meguro-ku
Tokyo, Japan

T. Yoshida
Dept. Inform. Processing
Tokyo Inst. Technology
227 Nagatsuta, Midori-ku
Yokohama, Japan

M. Sablatash
Communications Res. Centre
3701 Carling Ave.
P.O. Box 11490, Station H
Ottawa, Ontario, Canada

ABSTRACT

In this paper design methods for regular multidimensional perfect reconstruction (PR) filter banks are described. A systematic method for two-channel and four-channel filter banks is presented. The main novelties are: (1) the filters have impulse response with square support, rather than diamond support; (2) regular designs that have rectangular support are also presented, which are highly efficient in practice, since expensive memory is saved; (3) in addition to new diamond filters for the two-channel case, hexagonally-symmetric filters are also derived; and (4) novel 3-D filter banks are also designed. In all cases the filters are linear phase, achieve arbitrarily high regularity and can be used to obtain biorthogonal wavelet bases. The filter banks can be implemented in a structurally perfect-reconstruction manner.

1. INTRODUCTION

When 2-D and 3-D signals, such as still images and video, are coded using filter banks, linear phase of the individual filters is important. We aim at the design of linear-phase FIR filters, which also possess the regularity property. Regularity is a new requirement for filter banks imposed by the wavelet theory. The 2–D lowpass filter is regular when it has at least one zero at the point (π, π). The iteration of a regular filter bank leads to limit functions (scaling function and wavelet(s)). It is desireable to obtain smooth limit functions, possibly differentiable. It turns out that regularity can be important in practical applications like still image and video coding and compression, since regular filters provide smooth image representation [3]. Hence the problem is to achieve simultaneously the properties: PR + linear phase + non-separability + regularity. In this work we shall present several solutions to this problem.

2. THE 1–D CASE

We start our discussion with the 1–D case because: (1) it is the simplest starting point and (2) the derived filter banks will be used later. The perfect reconstruction (PR) conditions are

$$G_0(z) = H_1(-z) \tag{1}$$

$$G_1(z) = -H_0(-z) \tag{2}$$

and

$$H_0(z)H_1(-z) - H_1(z)H_0(-z) = 2z^{-l}, \tag{3}$$

where l must be odd. Suppose now that H_0 is a half-band linear-phase FIR filter of order $2N$, where N is odd, $H_0(z) = \sqrt{2}(H_{00}^N(z^2) + 0.5z^{-N})$. There are infinitely many PR filter pairs, (H_0, H_1), corresponding to [6]:

$$H_1(z) = \sqrt{2}z^{-N-M} - 2H_{00}^M(z^2)H_0(z). \tag{4}$$

Then H_1 will be of length $2M + 2N + 1$. To have maximum number of derivatives equal to zero at $z = -1$ we shall choose as a lowpass filter

$$H_0^N(\omega) = \sum_{i=0}^{(N-1)/2} \binom{N}{i} x^i (1-x)^{N-i}, \tag{5}$$

where $x = \sin^2(\omega/2)$. This $H_0(z)$ is divisible by $(1 + z^{-1})^N$ and it can be shown that H_1 is divisible by $(1 - z^{-1})^{\min(N,M)}$.

3. 2-D TWO-CHANNEL FILTER BANKS

3.1. Filters with square support

We start with the two-dimensional two-channel quincuncial case. In this case the PR conditions are

$$G_0(z_1, z_2) = H_1(-z_1, -z_2) \tag{6}$$

$$G_1(z_1, z_2) = -H_0(-z_1, -z_2) \tag{7}$$

$$H_0(z_1, z_2)H_1(-z_1, -z_2) - H_1(z_1, z_2)H_0(-z_1, -z_2) = 2z_1^{-2k_1+1}z_2^{-2k_2}. \tag{8}$$

If H_0 is a $2N+1 \times 2N+1$ half-band diamond-shaped zero-phase FIR filter

$$H_0^N(z_1, z_2) = 0.5 + z_1 H_{01}^N(z_1 z_2, z_1 z_2^{-1}) \tag{9}$$

Then we can meet the PR condition (8) easily by using

$$H_1 = -2z_1^{-2k_1+1}z_2^{-2k_2} - 2H_{01}^M(z_1 z_2, z_1 z_2^{-1})H_0. \tag{10}$$

The pair (H_0, H_1) can be implemented in such a way that the PR property is preserved even under finite-word-length arithmetic (Fig. 1). Filters having at least one zero at (π, π) are regular. For filters of support $2N+1 \times 2N+1$ the maximum number of zeros is $N - 1$.

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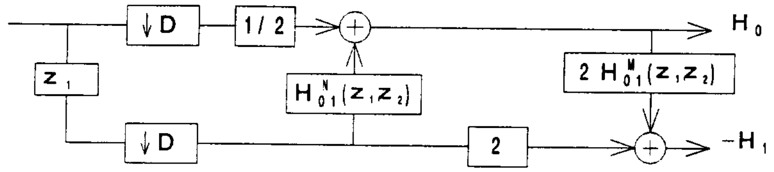

Fig. 1 Preserving the PR property implementation

3.1.1. Maximum number of vanishing moments

Using the bivariate Bernstein polynomial we can obtain filters with maximum number of vanishing moments.

$$H_0(z_1, z_2) = \frac{1}{2^{4N}} \sum_{i=0}^{N} \sum_{j=0}^{N-i} g_{i,j} \binom{N}{i} \binom{N}{j} (1 - z_1^{-1})^{2i}$$

$$(-1)^{i+j}(1 + z_1^{-1})^{2(N-i)}(1 - z_2^{-1})^{2j}(1 + z_2^{-1})^{2(N-j)},$$

where $g_{i,j} = 1$ when $i+j < N$ and $g_{i,j} = 0.5$ when $i+j = N$. Then H_0^N has a zero of order $N-1$ at (π, π) and H_1^M, obtained from (10), has a zero of order $M-1$ at $(0,0)$.

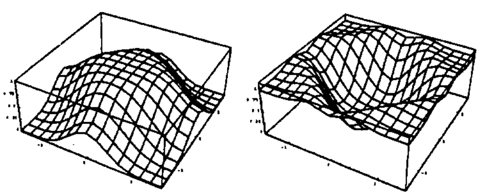
Fig. 2 Frequency responses of H_0 and H_1 ($N = M = 3$).

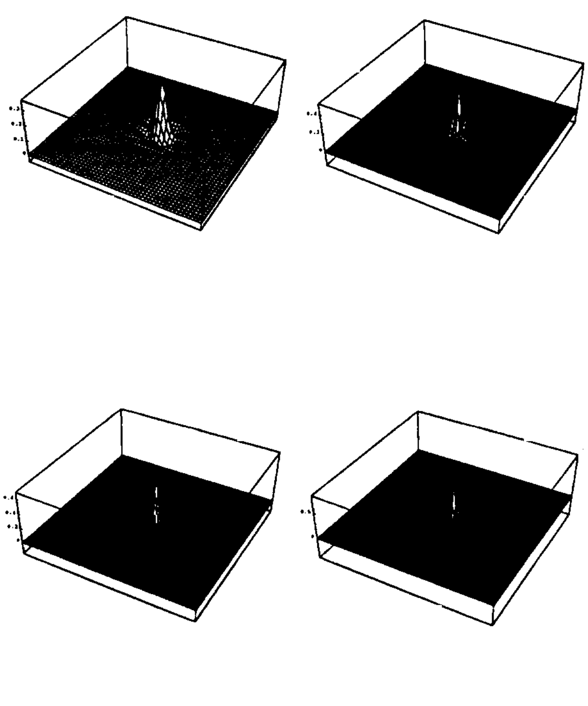
Fig. 3 Analysis and synthesis scaling functions and wavelets ($N = M = 5$).

3.1.2. Improved frequency response

Now we advance a new method that will ensure some regularity, linear-phase, structurally perfect-reconstruction implementation and improved frequency responses. Here we are going to use the Bernstein form of a bivariate polynomial

$$H(x,y) = \sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} \binom{N}{i} \binom{N}{j} x^i (1-x)^{N-i} y^j (1-y)^{N-j}. \tag{11}$$

Everything is controlled by the coefficients b_{ij}.

$$b_{ij} = \begin{cases} 1 & i+j < L-1 \\ 1 - \alpha_{i+j-L-1} & L \le i+j \le N-1 \\ 0.5 & i+j = N \\ \alpha_{2N-L-1-i-j} & N+1 \le i+j \le 2N-L \\ 0 & 2N-L-1 \le i+j \le 2N \end{cases}$$

Note that α is a one-dimensional array and whatever values the coefficients α_k have H_0 not only will always be half-band, but will have a zero of order L at $\omega_1 = \omega_2 = \pi$. Therefore the regularity constraint is exactly satisfied. The solution with maximum number of vanishing moments corresponds to $\alpha_i = 0$ for $i = 1, \cdots, N-L$. Here we shall use these $N-L$ additional degrees of freedom to improve the frequency response in the least-squares sense.

$$H(x,y) = \sum_{i=0}^{N} \sum_{j=0}^{N-i-1} \binom{N}{i} \binom{N}{j} (1-x)^{N-i} (1-y)^{N-j}$$

$$x^i y^j + \frac{1}{2} \sum_{i=0}^{N} \binom{N}{i} \binom{N}{i} x^i (1-x)^{N-i} y^{N-i} (1-y)^i$$

$$+ \sum_{k=1}^{N-L} \alpha_k \Big[\sum_{i=N-L-k+1}^{N} \binom{N}{i} \binom{N}{2N-L-k+1-i}$$

$$x^i (1-x)^{n-i} y^{2N-L-k+1-i} (1-y)^{-N+L+k-1+i}$$

$$- \sum_{i=0}^{L+k-1} \binom{N}{i} \binom{N}{L+k-1-i} x^i (1-x)^{n-i}$$

$$y^{L+k-1-i} (1-y)^{N-L-k+1+i} \Big] \tag{12}$$

It is clear that we can write $H(x,y) = cv^t(x,y)$, where c and $v(x,y)$ are vectors

$$c = [1 \quad \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{N-L}]$$

$$v_1(x,y) = \sum_{i=0}^{N} \sum_{j=0}^{N-i-1} \binom{N}{i} \binom{N}{j} (1-x)^{N-i} (1-y)^{N-j}$$

$$x^i y^j + \frac{1}{2} \sum_{i=0}^{N} \binom{N}{i} \binom{N}{i} x^i (1-x)^{N-i} y^{N-i} (1-y)^i$$

$$v_k(x,y) = \sum_{i=N-L-k+1}^{N} \binom{N}{i}\binom{N}{2N-L-k+1-i}$$
$$x^i(1-x)^{n-i}y^{2N-L-k+1-i}(1-y)^{-N+L+k-1+i}$$
$$-\sum_{i=0}^{L+k-1}\binom{N}{i}\binom{N}{L+k-1-i}x^i(1-x)^{n-i}y^{L+k-1-i}$$
$$(1-y)^{N-L-k+1+i} \qquad k=2,\cdots,N-L$$

Now we can find the coefficients α_i so that the stop-band energy of the filter $H(x,y)$ is minimum in the least-squares sense.

$$E_s = \int\int_{D_s} H^2(x,y)ds \tag{13}$$

where the area of integration is the stopband of the filter. For diamond-shaped filters the area is triangle.

$$E_s = c \ P \ c^T . \tag{14}$$

Because of the half-band property there is no need to include the passband error. Here $\omega_{1s} = \omega_{2s} = \omega_s$, since the filter is quadrantally symmetric and $v(x,y) = v(\sin^2(\omega_1/2), \sin^2(\omega_2/2))$. The element p_{mn} , $m,n = 1,\cdots,N-L+1$, of the matrix P is the integral of the outer product

$$p_{mn} = \int_{\omega_s}^{\pi} v^T(m)(\omega,\pi+\omega_s-\omega)\cdot v(n)(\omega,\pi+\omega_s-\omega)d\omega. \tag{15}$$

The eigenvector, corresponding to the minimum eigenvalue of the positive-definite matrix P is the solution.

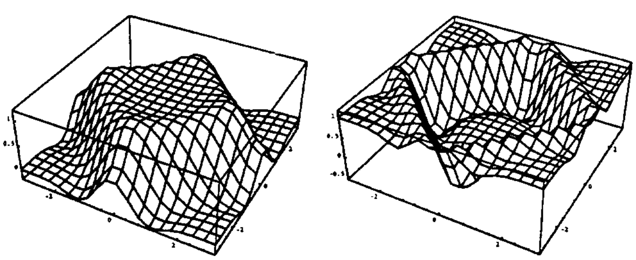

Fig. 4 Frequency responses of the regular PR filter pair H_0 and H_1, having sharp frequency responses.

3.2. Filters with rectangular support

In 2-D signal processing either z_1 or z_2 is much more expensive than the other. We can assume that z_1 represents line memory and is much more expensive than z_2. Therefore we can save memory if we make the support of the filter rectangular with $N_1 < N_2$. Note that any attempt to design the filter bank using transformation is doomed to failure. The first step is to design the filter H_0 having rectangular support and zeros at (π,π) [9]. The second step is to obtain H_1 by (10).

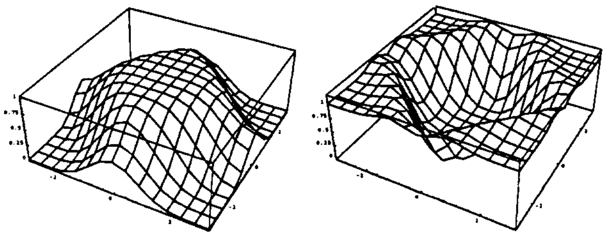

Fig. 5 Frequency responses of H_0 (5×7) and H_1 (9×13).

3.3. Filters with hexagonally-shaped frequency response

Another major difference between 1-D and 2-D multirate digital filters is that in the 2-D case we can have several filter passbands for the same downsampling matrix. This has been pointed out in [1], but this fact is not exploited in any of the previously available design techniques. In other words, for the same two-channel filter bank, assuming the same downsampling matrix, the diamond-shaped passband is not the only possible passband! Another choice would be the hexagonal shape. (This is different from 4-channel filter banks with hexagonally-symmetric filters [8]). Naturally we require a zero of arbitrary high order, specified in advance, at the point (π,π). Again the procedure consists of two steps. First, H_0 is designed as a a half-band filter with hexagonally-shaped frequency response [10]. Then H_1 is derived using (10).

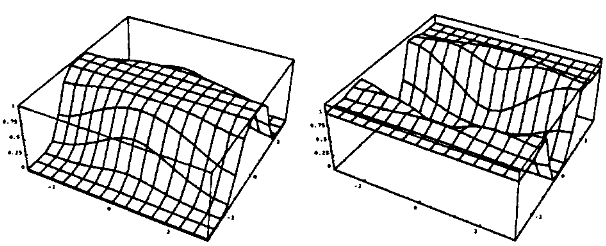

Fig. 6 Frequency responses of H_0 (9×15) and H_1 (17×29) having hexagonally-shaped frequency responses.

4. 2-D FOUR-CHANNEL FILTER BANKS

In the four-channel case the downsampling matrix has determinant equal to 4 and can be factored into two other matrices: the first represents separable subsampling and the second represents the quincuncial subsampling. First a one-dimensional filter bank is used, and then a two-dimensional filter bank for quincuncial downsampling. The result is a four-channel filter bank where hexagonal downsampling is performed. Thus we must design two PR filter banks, which we already did in the previous sections. Note that the 2-D filter bank could be composed of filters having square, rectangular or hexagonal support of the impulse response. We give an example where the filters have rectangular support.

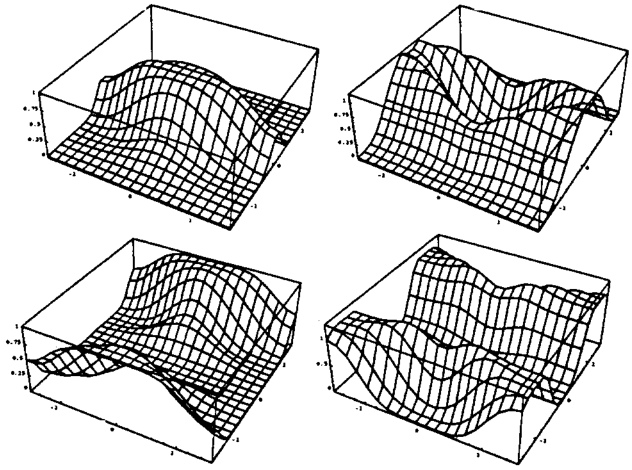

Fig. 7 Frequency responses of the four equivalent filters of the four-channel filter bank. The lowpass and highpass filters of the 1-D filter bank are of length 7 and 13 and the filters of the 2-D filter bank are 7 × 7 and 13 × 13, respectively.

5. 3–D FILTER BANKS

There are two important special cases: face-centered (FC) downsampling and body-centered (BC) downsampling.

5.1. 3–D two-channel filter banks

To achieve PR, first aliasing is avoided by specifying

$$G_0(z_1, z_2, z_3) = H_1(-z_1, -z_2, -z_3) \quad (16)$$
$$G_1(z_1, z_2, z_3) = -H_0(-z_1, -z_2, -z_3), \quad (17)$$

and second

$$H_0^N(z_1, z_2, z_3) = 0.5 + z_1 H_{01}^N(z_1, z_2, z_3) \quad (18)$$
$$H_1^M(z_1, z_2, z_3) = -2z_1 - 2H_{01}^M(z_1, z_2, z_3)H_0^N. \quad (19)$$

Again the Bernstein polynomial provides a solution with maximum number of vanishing moments:

$$H_0^N(z_1, z_2, z_3) = \frac{1}{2^{6N}} \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N-i-j} g_{i,j,k} \binom{N}{i} \binom{N}{j}$$
$$\binom{N}{k} (-1)^{i+j+k} (1 - z_1^{-1})^{2i} (1 + z_1^{-1})^{2(N-i)} (1 - z_2^{-1})^{2j}$$
$$(1 + z_2^{-1})^{2(N-j)} (1 - z_3^{-1})^{2k} (1 + z_3^{-1})^{2(N-k)}, \quad (20)$$

where $g_{i,j,k} = 1$ for $i + j + k < N$ and $g_{i,j,k} = 0.5$ when $i + j + k = N$. H_1 is obtained again from (10). When $M = N$ the analysis and synthesis filters have equal numbers of vanishing moments. In a similar way, as for the 2-D case, the support of the impulse response can have the shape of a parallelepiped. The extension to the N-dimensional case is now straightforward.

5.2. 3–D four-channel filter banks

The PR conditions are more lengthy and are not included here. The BC and FC type of downsampling are related in much the same way as the hexagonal and quincunx downsampling in the 2–D case. We can construct the filter bank as a cascade connection of two 2-channel filter banks.

6. CONCLUSIONS

In this paper regular multidimensional perfect reconstruction filter banks are studied. Several cases, which are important for applications, are considered. All filters in this paper have linear phase and an arbitrarily high number of vanishing moments. The method, presented here is the only one, that is characterized by square or cubic support. An efficient implementation of the filter bank exists, which preserves the PR property even when the finite-word-length effects are considered [5].

A much more detailed description will be presented in another publication.

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