OVERSAMPLED MODULATED FILTER BANKS AND TIGHT GABOR FRAMES IN $\ell^2(\mathbf{Z})$

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ABSTRACT

The subject of this study is paraunitary [1] modulated filter banks. A factorization of the polyphase matrices of these filter banks, which is described here, gives complete characterization of tight Gabor frames in $\ell^2(\mathbf{Z})$, with arbitrary rational oversampling ratios. Tight Gabor frames, being less constrained than orthogonal bases, allow for filter bank designs with good localization in both time and frequency.

1. Introduction

Gabor functions have been shown to be useful for analysis of continuous time signals, owing to their local character in the joint time-frequency domain. Along with the development of Gabor's original scheme in the 1980's, it was observed that such local time-frequency representations can be stable only in the overcomplete case [2, 3]. Another incarnation of the same phenomenon is expressed by the Balian-Low theorem [4, 5] which implies that there are no orthogonal Gabor bases which have good localization in both time and frequency. However, as soon as some redundancy is introduced, the picture changes drastically and, as demonstrated by Daubechies [6], good localization of tight Gabor frames is attainable.

In $\ell^2(\mathbf{Z})$ Gabor frames are equivalent to modulated filter banks. An effect similar to that described by the Balian-Low theorem has been observed by Vetterli [7], who demonstrated that there are no critically sampled modulated filter banks with finite impulse responses and good frequency selectivity. The purpose of this paper is to investigate the existence of oversampled modulated filter banks, in particular those which generate tight frames, with finite support in time and also good localization in frequency.

In the following, a complete characterization of tight Gabor frames in $\ell^2(\mathbf{Z})$ with arbitrary rational oversampling ratios is described within the framework of modulated filter banks. Oversampled filter banks are much less constrained than critically sampled ones and consequently allow for the design of frequency selective FIR linear phase filters. In other words, there exist tight Gabor frames in $\ell^2(\mathbf{Z})$ with symmetries and good localization in the time-frequency domain. We provide a design example to illustrate this point.

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2. Polyphase Description of Oversampled Modulated Filter Banks

We investigate the properties of oversampled filter banks using polyphase domain analysis [1]. Analysis filters of a K-channel filter bank are denoted by $G_0(z),\ G_1(z)\ldots G_{K-1}(z)$ while the associated polyphase matrix is denoted by $\mathbf{G}_p(z)$. If decimation by a factor of $N \leq K$ is employed in each channel, $\mathbf{G}_p(z)$ is a $K\times N$ matrix of polyphase components, $[\mathbf{G}_p(z)]_{ij} = G_{ij}(z)$ is the j-th polyphase component of $G_i(z)$.

In the case of modulated filter banks, the analysis filters are obtained from a single prototype lowpass filter, $H(z)$, according to $G_i(z) = H(W_K^i z)$, where $W_K = exp(j\frac{2\pi}{K})$. Let M be the least common multiple of K and N and let J and L be the two integers satisfying $JK = LN = M$. The M-component polyphase representation of $H(z)$ has the form:

$$H(z) = \sum_{j=0}^{M-1} z^{-j} H_j(z^M). \qquad (1)$$

Elements of the polyphase matrix can be expressed as

$$G_{ij}(z) = \sum_{l=0}^{L-1} W_K^{-i(j+lN)} z^{-l} H_{j+lN}(z^L). \qquad (2)$$

This gives following factorization:

$$\mathbf{G}_p(z) = \mathbf{W}_K \mathbf{B}(z), \qquad (3)$$

where $\mathbf{W}_K$ is the $K \times K$ DFT matrix and

$$\mathbf{B}(z) = [I_K ... I_K] \cdot$$
$$\cdot \text{diag}\left(H_0(z^L) ... H_{M-1}(z^L)\right) \begin{bmatrix} I_N \\ z^{-1} I_N \\ \cdots \\ z^{-(L-1)} I_N \end{bmatrix}$$

I_n here stands for the $n \times n$ identity matrix.

By inspection of the above factorization one can see that elements of $\mathbf{B}(z)$ are given by

$$[\mathbf{B}(z)]_{ij} = z^{-q} H_{i+pK}(z^L) \quad \begin{array}{l} i = 0, 1, ..., K-1, \\ j = 0, 1, ..., N-1 \end{array} \qquad (4)$$

where p and q are integers satisfying

$$i + pK = j + qN, \quad p \leq J - 1, \quad q \leq L - 1. \qquad (5)$$

Note that equation (5) cannot be satisfied for every pair of integers i and j. In fact, for each j there are exactly J indices, i, which satisfy (5). Consequently J nonzero elements are evenly distributed in each row of $\mathbf{B}(z)$ and L nonzero elements are evenly distributed in each column of $\mathbf{B}(z)$. Three possible cases are illustrated by following examples.

Example 1 K is a multiple of N. In this case, $J = 1$, so that there is a single nonzero element in each row of $\mathbf{B}(z)$. For $K = 6$ and $N = 3$ we have:

$$\mathbf{B}(z) = \begin{bmatrix} H_0(z^2) & 0 & 0 \\ 0 & H_1(z^2) & 0 \\ 0 & 0 & H_2(z^2) \\ z^{-1}H_3(z^2) & 0 & 0 \\ 0 & z^{-1}H_4(z^2) & 0 \\ 0 & 0 & z^{-1}H_5(z^2) \end{bmatrix}.$$

Example 2 K and N are coprime. In this case, $J = N$, so that all elements of $\mathbf{B}(z)$ are nonzero. For $K = 3$ and $N = 2$ we obtain:

$$\mathbf{B}(z) = \begin{bmatrix} H_0(z^3) & z^{-1}H_3(z^3) \\ z^{-2}H_4(z^3) & H_1(z^3) \\ z^{-1}H_2(z^3) & z^{-2}H_5(z^3) \end{bmatrix}$$

Example 3 Neither of the above cases, eg. $K = 6$ and $N = 4$ yields $\mathbf{B}(z)$ equal to the following matrix:

$$\begin{bmatrix} H_0(z^3) & 0 & z^{-1}H_6(z^3) & 0 \\ 0 & H_1(z^3) & 0 & z^{-1}H_7(z^3) \\ z^{-2}H_8(z^3) & 0 & H_2(z^3) & 0 \\ 0 & z^{-2}H_9(z^3) & 0 & H_3(z^3) \\ z^{-1}H_4(z^3) & 0 & z^{-2}H_{10}(z^3) & 0 \\ 0 & z^{-1}H_5(z^3) & 0 & z^{-2}H_{11}(z^3) \end{bmatrix}$$

3. Tight Gabor Frames in $\ell^2(\mathbf{Z})$

Tight frame conditions on oversampled filter banks are given by the following proposition, which holds for obvious reasons.

Proposition 1 *An oversampled filter bank implements a tight frame decomposition in $\ell^2(\mathbf{Z})$ if and only if its polyphase analysis matrix is paraunitary.*

According to the definitions of $\mathbf{B}(z)$ and $\mathbf{G}_p(z)$, the paraunitariness condition [1] on $\mathbf{G}_p(z)$ is equivalent to

$$\tilde{\mathbf{B}}(z)\mathbf{B}(z) = cI_N.$$

For each element $\left[\tilde{\mathbf{B}}(z)\mathbf{B}(z)\right]_{ij}$ this condition imposes the constraint:

$$\sum_{l=0}^{L-1} \tilde{H}_{i+lN}(z^L) H_{i+lN}(z^L) = c, \text{ if } i = j, \qquad (6)$$

or

$$\sum_{l=0}^{L-1} z^{-\sigma(q+l)} \tilde{H}_{i+lN}(z^L) H_{i+pK+lN}(z^L) = 0, \text{ if } i \neq j. \qquad (7)$$

In (7), p and q have the same interpretation as in (5) and the function, $\sigma(\cdot)$, is given by:

$$\sigma(q+l) = \begin{cases} q, & q+l < L \\ -(L-q), & q+l \geq L \end{cases}$$

Noting that some of elements of $\tilde{\mathbf{B}}(z)\mathbf{B}(z)$ are identically zero and taking symmetries into account, the paraunitariness condition imposes $N + \frac{N(J-1)}{2}$ distinct constraints, N of which are given by (6), while the rest are given by (7). We now investigate the implications of these constraints for the three cases presented in Section 2.

Case 1 *K is a multiple of N.*
Here, $H(z)$ is constrained only by (6), which indicates that the polyphase components

$$\{H_i(z), H_{i+N}(z), ..., H_{i+(L-1)N}(z)\},$$

for each $i = 0, ..., N-1$ are power complementary. It follows that they are the polyphase components of a filter $F_i(z)$, which is orthogonal to its translates by multiples of L. Hence tight Gabor frames with an integer oversampling ratio K/N are generated by arbitrarily selecting N orthogonal filters $F_i(z)$ and identifying their polyphase components with the polyphase components of $H(z)$.

Case 2 *K and N are coprime*
In this case, the polyphase components of $H(z)$ are, up

to a time delay, equal to the entries of a full $K \times N$ paraunitary matrix. For instance, in Example 2, paraunitariness of $\mathbf{G}_p(z)$ is equivalent to paraunitariness of the matrix

$$\mathbf{B}_0(z) = \begin{bmatrix} H_0(z) & H_3(z) \\ z^{-1}H_4(z) & H_1(z) \\ H_2(z) & H_5(z) \end{bmatrix}.$$

Case 3 *Neither of the above cases*
The polyphase components of $H(z)$ are again, up to a time delay, simply the entries of N/J paraunitary matrices, each of dimension $L \times J$. In Example 3 of Section 2, $\mathbf{B}(z)$ is parauntiry if and only if the matrices

$$\mathbf{B}_0(z) = \begin{bmatrix} H_0(z) & H_6(z) \\ z^{-1}H_8(z) & H_2(z) \\ H_4(z) & H_{10}(z) \end{bmatrix}$$

and

$$\mathbf{B}_1(z) = \begin{bmatrix} H_1(z) & H_7(z) \\ z^{-1}H_9(z) & H_3(z) \\ H_5(z) & H_{11}(z) \end{bmatrix}$$

are paraunitary. Hence, finding $\mathbf{B}(z)$ amounts to finding the two 3×2 paraunitary matrices, $\mathbf{B}_0(z)$ and $\mathbf{B}_1(z)$.

Conditions (6) and (7), being both necessary and sufficent, represent a complete parametrization of tight Gabor frames in $\ell^2(\mathbf{Z})$. Design of of the tight frames then amounts to an optimization procedure under these constraints. Recall that, in the case of perfect reconstruction FIR modulated filter banks with critical sampling, the polyphase components of $H(z)$ can not have more than one nonzero coefficient [7], which is too restrictive to obtain good frequency selectivity. We show, by the following design example, that in the oversampled case tight frames with good time frequency localization are attainable.

Example 4 Consider case the $N = 2$ and $K = 4$. With the additional requirement that the prototype filter $H(z)$ is symmetric, the design consists of finding a single filter $F(z)$ which is orthogonal to its translates by multiples of 2. In terms of their polyphase components, $F(z)$ and $H(z)$ are given by

$$F(z) = F_0(z^2) + z^{-1}F_1(z^2)$$
$$H(z) = H_0(z^4) + z^{-1}H_1(z^4) + z^{-2}H_2(z^4) + z^{-3}H_3(z^4)$$

The design constraints are thus satisfied by taking $H_0(z)$ and $H_2(z)$ to be equal to $F_0(z)$ and $F_1(z)$, respectively, and $H_1(z)$ and $H_3(z)$ to be their time reversed versions. One solution, obtained from a 4-tap filter $F(z)$, is shown in Figure 1. Of course, better frequency selectivity could be achieved with longer filters, $F(z)$

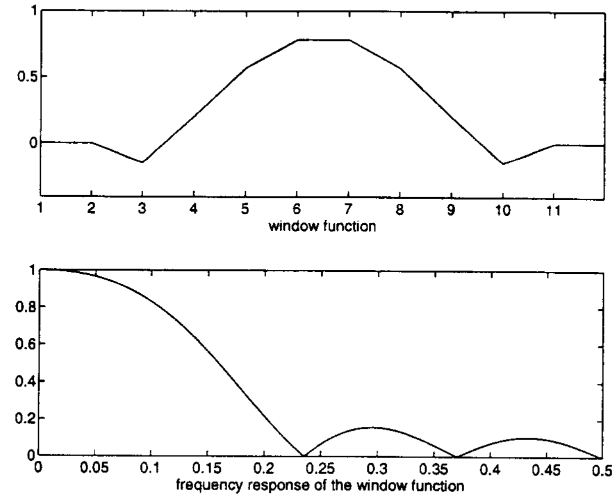

Figure 1: *An example of an 8-tap window function for tight Gabor frames with the oversampling factor $K/N = 2$.*

4. Conclusion

In this paper, we have given a complete parametrization of tight Gabor frames in $\ell^2(\mathbf{Z})$. It turns out that in the oversampled case there is sufficient freedom for the design of tight Gabor frames with good localization in the time-frequency domain.

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