EXACT RECONSTRUCTION FROM PERIODIC NONUNIFORM SAMPLES

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ABSTRACT

We examine the problem of reconstructing a discrete-time signal given only n of its M-phase components. Borrowing analysis from the field of perfect reconstruction filter banks enables us to derive necessary and sufficient conditions under which reconstruction is possible. Essentially, in a perfect reconstruction system, the conditions to reconstruct from partial information are equivalent to the conditions to ensure that the rest of the information does not contribute to the reconstructed signal.

An application is that this allows us to reconstruct multiband signals which have overall bandwidth of no more than B, yet cannot be reconstructed from uniformly spaced samples at the minimum rate $B/2\pi$.

1 INTRODUCTION

It is well known that if a signal is lowpass bandlimited to the region $|\Omega| < B/2$ then it can be exactly reconstructed from samples uniformly spaced $B/2\pi$ apart. However, if a signal covers the same amount B of the spectrum, but is bandpass rather than lowpass the same is not *necessarily* true. It depends on whether the aliased versions of the spectrum $X(\Omega + rB)$ overlap or not. If any overlap does occur then reconstruction using *uniformly spaced* samples at the rate $B/2\pi$ is not possible. For a bandpass signal the minimum rate needed if uniform sampling is used can vary from $B/2\pi$ to B/π depending on the location of the band edges. A detailed treatment is given in [8].

If the signal is multiband then the situation is even more involved. If the total bandwidth is again B, it is known that an average sampling rate of at least $B/2\pi$ will be required, which is known as the Nyquist-Landau rate [5]. Again, uniform sampling at this rate is possible only if overlap between the spectral replicas does not occur [3]. For a multiband signal with arbitrary bands, the minimum rate at which uniform sampling can be used is quite unpredictable. On the other hand second order sampling [4], or use of nonuniform periodic sampling can allow a lower average rate for bandpass and multiband signals [2, 6, 7]. The conditions under which nonuniform sampling could be used has not been thoroughly understood however.

Part of the work will be to establish how familiar tools from multirate filter banks can be used to solve this problem. While we work in the discrete-time case for multiband signals, the results that we generate also apply to nonuniform periodic sampling of continuous-time signals. We examine the conditions under which a multiband signal can be reconstructed using periodic nonuniformly spaced samples. For example if $X(e^{j\omega})$ is limited to the range $(-3\pi/6, -\pi/6) \cup (\pi/6, 3\pi/6)$ it cannot be reconstructed from every third sample, as shown in Figure 2 (a), even though it's spectrum occupies only $1/3$ of the region $(-\pi, \pi)$ and thus two thirds of the samples are redundant. However, it can be reconstructed from nonuniform samples, with the same overall rate, as shown in Figure 2 (b).

2 RECONSTRUCTION OF A SIGNAL FROM N OF ITS M-PHASE COMPONENTS

Suppose that a discrete-time signal $X(z)$ is written in terms of its M-phase components

$$X(z) = X_0(z^M) + z^{-1}X_1(z^M) + \cdots z^{-(M-1)}X_{M-1}(z^M).$$

We may write the problem of nonuniform sampling as that of reconstructing $X(z)$ from only n of the components $X_i(z)$. This implies that (at least) $M - n$ out of every M samples are redundant. This is different to the problem of reconstruction from uniformly spaced samples, since we now have n uniformly spaced sample trains, separated by known phases. The rate is n/M. An example of the difference is shown in Figure 2 (a) where a every third sample of a sequence is retained, and Figure 2 (b) where every sixth sample shifted by 0 and shifted by 2 is retained. The rate in both cases is the same, but the sampling pattern and the conditions under which the signal can be reconstructed are in general very different.

To determine these conditions consider the structure shown in Figure 1, which is an M-channel perfect reconstruction filter bank. We split the signal using a set of analysis filters $H_0(z), H_1(z), \cdots H_{M-1}(z)$ and recombine using the synthesis filters $G_0(z), G_1(z), \cdots G_{M-1}(z)$. Just as we did for $X(z)$ we can write a filter in terms of its M-phase components

$$H_i(z) = H_{i0}(z^M) + z^{-1}H_{i1}(z^M) + \cdots z^{-(M-1)}H_{iM-1}(z^M).$$

Often a multirate filter bank is written in terms of its polyphase matrices

$$[\mathbf{H}_p(z)]_{ij} = H_{ij}(z), \quad \text{and} \quad [\mathbf{G}_p(z)]_{ij} = G_{ij}(z).$$

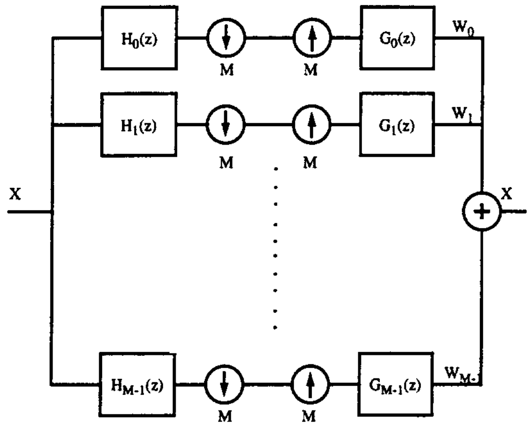

Figure 1: Maximally decimated M-channel multirate filter bank. If n of the analysis filters are simple delays, this structure reconstructs the signal from only n polyphase components, provided the other $M-n$ channels can be forced to zero.

It can be shown that the system is perfectly reconstructing if

$$\mathbf{G}_p^T(z) \cdot \mathbf{H}_p(z) = \mathbf{I}. \quad (1)$$

A trivially simple solution is the case where $\mathbf{G}_p^T(z) = \mathbf{H}_p(z) = \mathbf{I}$ which is known as the polyphase decomposition, since it merely separates the signal into it's M-phase components. In this case the i-th channel carries $X_i(z)$. This separation will be useful since we wish to reconstruct from some collection of n of the M-phase components. Denote the set containing the indices of these n components by A, and the set of the indices of the missing components by B; clearly then $A \cup B = \{0, 1, 2, \cdots M-1\}$. In other words we wish reconstruct $X(z)$ from the $X_i(z), i \in A$; so we want

$$X(z) = \sum_{i \in A} X_i(z^M)\lambda_i(z), \quad (2)$$

for some filters $\lambda_i(z)$.

Consider if, instead of the straightforward polyphase decomposition, *i.e.* using $\mathbf{H}_p(z) = \mathbf{I}$, we use a modified polyphase matrix $\mathbf{H}_p(z) = (\mathbf{I} - \mathbf{A})$, where the matrix $\mathbf{A}$ has the form

$$a_{ik}(z) = 0, i \in A \quad (3)$$
$$a_{kj}(z) = 0, j \in B. \quad (4)$$

This means that row i of $\mathbf{A}$ is zero if $i \in A$, and column j is zero if $j \in B$. This ensures that $\mathbf{A}^2 = \mathbf{0}$, independently of the choice of the $a_{ij}(z)$, since

$$\sum_{k=0}^{M-1} a_{ik}(z) \cdot a_{kj}(z) = 0 \ \ \forall\ i, j.$$

Thus, we can identify $\mathbf{G}_p^T(z) = (\mathbf{I} + \mathbf{A})$ as the inverse of $\mathbf{H}_p(z) = (\mathbf{I} - \mathbf{A})$, since $(\mathbf{I}+\mathbf{A}) \cdot (\mathbf{I}-\mathbf{A}) = \mathbf{I} + \mathbf{A} - \mathbf{A} + \mathbf{A}^2$.

The result of these manipulations is as follows. The analysis filters have the following form

$$H_i(z) = \begin{cases} z^{-i} & i \in A \\ z^{-i} - \sum_{k \in A} a_{ik}(z^M) z^{-k} & i \in B \end{cases} \quad (5)$$

And the synthesis filters can be expressed

$$G_i(z) = \begin{cases} z^{i} + \sum_{k \in B} a_{ik}(z^M) z^{k} & i \in A \\ z^{i} & i \in B \end{cases} \quad (6)$$

Notice that, while $M-n$ of the filters on the analysis side are modified with respect to the polyphase decomposition, only n are changed on the synthesis side.

An example may help to clarify matters. Suppose we choose $n = 2, M = 6$ and $A = \{0, 2\}$; thus we wish to reconstruct the signal from the zero-th and the second 6-phase components, as in Figure 2 (b). The form of the analysis polyphase matrix is then

$$\mathbf{H}_p(z) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -a_{10}(z) & 1 & -a_{12}(z) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -a_{30}(z) & 0 & -a_{32}(z) & 1 & 0 & 0 \\ -a_{40}(z) & 0 & -a_{42}(z) & 0 & 1 & 0 \\ -a_{50}(z) & 0 & -a_{52}(z) & 0 & 0 & 1 \end{bmatrix} = \mathbf{I} - \mathbf{A}. \quad (7)$$

The analysis filters are given by $H_i(z) = z^{-i},\ i \in \{0, 2\}$ and

$$H_i(z) = -a_{i0}(z^6) + z^{-i} - z^{-2} a_{i2}(z^6) \ \ i \in \{1, 3, 4, 5\}.$$

The corresponding synthesis matrix $\mathbf{G}_p^T(z)$ is $\mathbf{I} + \mathbf{A}$. The synthesis filters are then $G_i(z) = z^i,\ i \in \{1, 3, 4, 5\}$ and

$$G_0(z) = 1 + z a_{10}(z^6) + z^3 a_{30}(z^6) + z^4 a_{40}(z^6) + z^5 a_{50}(z^6)$$
$$G_2(z) = z^2 + z a_{12}(z^6) + z^3 a_{32}(z^6) + z^4 a_{42}(z^6) + z^5 a_{52}(z^6).$$

Returning to the general case, examine what (5) and (6) imply in the filter bank shown in Figure 1. Since (1) holds with the choice of filters above, we have perfect reconstruction. Because of (5) however, the n given polyphase components (whose indices are in A) pass unchanged through the analysis bank, and thus the input to synthesis filter $G_i(z)$ for $i \in A$ is simply $X_i(z^M)$. It follows that if we can somehow force the output of channels $i \in B$ to be zero (*i.e.* $W_i(z) = 0, i \in B$ in the figure) then we have reconstructed the signal exactly from only the n polyphase components $i \in A$, using the $G_i(z), i \in A$ as the interpolation filters.

Theorem 2.1 *Let A be a set containing n of the indices $\{0, 1, 2, \cdots M-1\}$. In order to reconstruct a signal $X(z)$ from the n given M-phase components $X_i(z), i \in A$ it is necessary and sufficient that there exist $M-n$ filters $H_i(z)$ such that*

$$H_i(e^{j\omega}) X(e^{j\omega}) = 0, \quad i \notin A,\ \forall\ \omega, \quad (8)$$

where each $H_i(z)$ is of the form

$$H_i(z) = z^{-i} - \sum_{k \in A} a_{ik}(z^M) z^{-k},\ i \notin A. \quad (9)$$

Proof: We first show that if the signal can be reconstructed from the given M-phase components, that is written in the form (2) for some filters $\lambda_i(z)$, that these can always have the form that we have assumed in (6). Consider the restriction that (6) implies for the polyphase componenets of $G_i(z)$ where $i \in A$ (the filters $i \in B$ play no role in the reconstruction (2)):

$$G_{ik}(z) = \begin{cases} a_{ik}(z) & k \in B, i \in A \\ \delta_{ik} & k \in A, i \in A \end{cases} \qquad (10)$$

Also consider the k-th polyphase component of the reconstruction (2)

$$X_k(z^M) = \sum_{i \in A} X_i(z^M)\lambda_{ik}(z^M). \qquad (11)$$

For the components $X_k(z)$ where $k \in B$ (10) is no restriction since the appropriate polyphase terms in the sum are arbitrary. For $k \in A$ it is clear that if we use the $G_{ik}(z)$ instead of the $\lambda_{ik}(z)$ in (11) this gives a solution. Hence the constrained form that we have assumed for the filters $G_i(z)$ poses no obstacle to the reconstruction.

Next, observe that (8) is sufficient: if each branch $i \in B$ is zero at the output of the analysis filters, then that branch, and that M-phase component $X_i(z)$ plays no role in the reconstruction. However (8) is also necessary, since if the signal is to be recovered from only the $X_i(z), i \in A$ we must have

$$\sum_{i \in B} W_i(z) = 0.$$

This implies that each of the $W_i(z)$ individually equals zero, since (6) gives that each $W_i(z)$ is of the form $W_i(z) = z^i f_i(z^M)$ for some $f_i(z)$. That is the $W_i(z)$ are at different phases, and no cancellation between them is possible. Thus we must have $W_i(z) = 0, i \in B$ which in turn gives (8).□

To achieve reconstruction of the signal, we must construct $M - n$ filters, of the form given in (9) which satisfy (8). In order for (8) to be satisified the spectrum of the filter must be zero for all frequencies where the spectrum of the signal is non-zero. Since we are dealing with discrete-time sequences all of the spectra are 2π-periodic, so denote that part of the region $(-\pi, \pi)$ where $X(e^{j\omega})$ differs from zero by S. It follows from (8) and (9) that we must have

$$e^{-j\omega i} = \sum_{k \in A} a_{ik}(e^{j\omega M})e^{-j\omega k}, \;\; i \notin A, \omega \in S. \qquad (12)$$

Designing the reconstruction scheme, now involves finding the $a_{ik}(e^{j\omega M})$ such that this holds over the required frequency bands.

For a fixed i, all of the $a_{ik}(e^{j\omega M})$ are $2\pi/M$-periodic, so we can fix the value of these over some set of frequencies which occupies no more than $2\pi/M$ of the frequency axis in total; the values elsewhere will then be fixed by periodicity. However, the constraint (12) has to be satisfied for all $\omega \in S$, and the set S possibly contains several frequencies seperated by integer multiples of $2\pi/M$, which may conflict with the periodicity requirement. If there were $n+1$ such frequencies in S, then the system could not be solved, since (12), evaluated at the $n+1$ frequencies, would gives a system of $n+1$ equations in n unknowns (the $a_{ik}(e^{j\omega_p M}), p = 0, 1 \cdots n-1$). This gives one limitation on the reconstruction: $X(e^{j\omega})$ cannot be non-zero at more than n frequencies separated by integer multiples of $2\pi/M$. Expressed another way, in the sum

$$\sum_{k=0}^{M-1} X(e^{j(\omega + 2\pi k/M)})$$

no more than n terms should be non-zero for any value of $\omega \in S$. We now show that if this is satisfied then reconstruction is always possible. Suppose that there are only n frequencies separated by $2\pi/M$; then (12) gives n equations in n unknowns. To consider an example, in the $n = 2, M = 6$ case already introduced above with $A = \{0, 2\}$ the system is

$$\begin{bmatrix} 1 & e^{-j\omega_0 2} \\ 1 & e^{-j(\omega_0 + 2\pi/6)2} \end{bmatrix} \cdot \begin{bmatrix} a_{i0}(e^{j\omega_0 6}) \\ a_{i2}(e^{j\omega_0 6}) \end{bmatrix} = \begin{bmatrix} e^{-j\omega_0 i} \\ e^{-j(\omega_0 + 2\pi/6)i} \end{bmatrix}$$

Whether or not the system has a solution however depends on the sampling phases (*i.e.* the choice of the n indices in the set A). In this case, the determinant is

$$\Delta = e^{-j\omega_0 2}(e^{-2\pi/3} - 1) \neq 0.$$

If we choose $A = \{0, 3\}$ however the determinant equals zero and it is not possible to solve for the $a_{ik}(e^{j\omega_0 6})$. Thus, even in the simple case of $n = 2$ there are considerable differences between different sampling strategies. For larger n, with arbitrary A, determining whether (12) has a solution or not may be difficult. For certain choices of A, a solution *always* exists however. For example, take $A = \{0, 1, 2, \cdots n-1\}$. In this case the matrix to be inverted becomes

$$\begin{bmatrix} 1 & e^{-j\omega_0} & e^{-j\omega_0 2} & \dots & e^{-j\omega_0(n-1)} \\ 1 & e^{-j\omega_1} & e^{-j\omega_1 2} & \dots & e^{-j\omega_1(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-j\omega_{n-1}} & e^{-j\omega_{n-1} 2} & \dots & e^{-j\omega_{n-1}(n-1)} \end{bmatrix}. \qquad (13)$$

This is easily recognised as a square Vandermonde matrix, and is always nonsingular since the $e^{-j\omega_p}$ are distinct. We can summarise the foregoing.

Theorem 2.2 *If the signal $X(e^{j\omega})$ is non-zero over a set S that occupies at most n/M of the frequency spectrum, and at no frequency in S are more than n terms in the sum $\sum_{k=0}^{M-1} X(e^{j\omega + 2\pi k/M})$ non-zero, then $X(e^{j\omega})$ can always be exactly reconstructed from some set of n of its M-phase components.*

The advantage of non-uniform sampling can be shown in the example we considered before, where $X(e^{j\omega})$ is limited to the range $(-3\pi/6, -\pi/6) \cup (\pi/6, 3\pi/6)$. Choosing $n = 2, M = 6$ we indeed find that the conditions of the theorem are satisfied and reconstruction using, for example, the first and second 6-phase components, gives a minimum rate sampling, while reconstruction using every third sample is not.

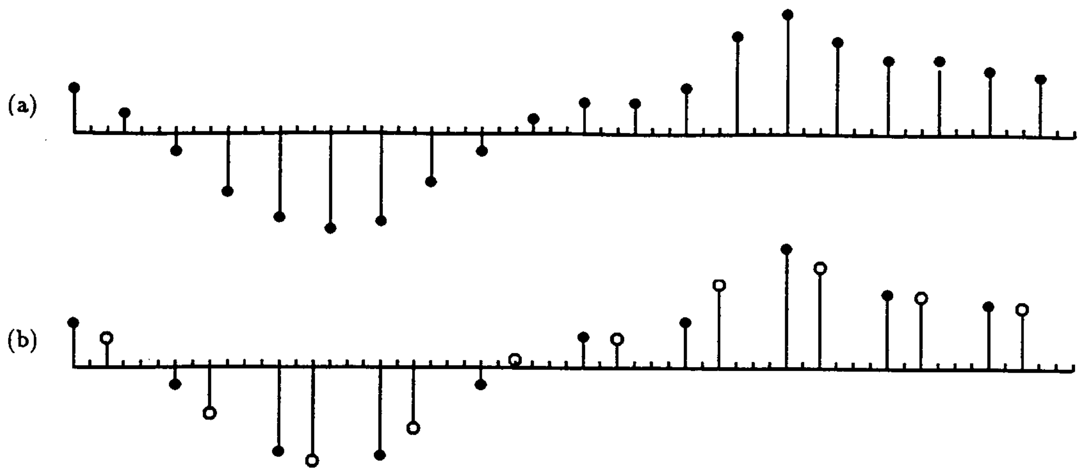

Figure 2: Sampling of a discrete-time sequence. (a) Uniform sampling where every third sample is retained. (b) Nonuniform sampling where every 6-th sample, shifted by zero and two are retained. Note that the rate is the same as in (a).

Remarks:

1. The derivation of Theorem 2.1 depended on the particular form that we chose for the polyphase matrices. All we then did was observe that, in a perfect reconstruction system, the conditions to reconstruct exactly from partial information are the same as the conditions to ensure that the rest of the information cannot contribute to the reconstructed signal. We solved the problem for the one dimensional case, but it should be clear that it works also for multidimensional systems.

2. When $n = 1$ the solutions can be used to design interpolators that have excellent convergence properties when used in iterative subdivision schemes [1].

3. We demonstrated that a solution can always be found by choosing $A = \{0, 1, 2, \cdots n-1\}$; this, however, may represent an undesirable solution since the samples are clustered together. This may make it worthwhile exploring other sets of indices A. Sampling schemes which also yield Vandermonde systems will also be given by, *e.g.* $A = \{0, 2, 4, \cdots 2n-2\}$.

4. If $X(e^{j\omega})$ is non-zero over a set that occupies only n/M of the interval $(-\pi, \pi)$, but does not satisfy the second requirement of Theorem 2.2, it is always possible to try $n'/M' \approx n/M$.

3 MINIMUM RATE SAMPLING OF CONTINUOUS-TIME MULTIBAND SIGNALS

In the last section we showed how to reconstruct discrete-time sequence with a multiband spectrum. We can apply these results to the problem of sampling multiband continuous-time signals. Suppose a real continuous-time multiband signal has an overall bandwidth of B, and the highest frequency at which it has non-zero energy is Ω_c. Clearly then we can uniformly sample the signal at $\Omega_s \geq 2\Omega_c$ without loss. The fraction of the discrete spectrum $(-\pi, \pi)$ that will be occupied is B/Ω_s. We can reconstruct this discrete signal $X_s(e^{j\omega})$ from n of its M-phase components if we can find n and M that satisfy Theorem 2.2. The first constraint, that $X_s(e^{j\omega})$ should occupy no more than n/M of the spectrum merely requires $n/M \geq B/\Omega_s$. A condition, then for sampling at the Nyquist-Landau rate, using non-uniform periodic samples is that we can find n, M and Ω_s such that $n/M = B/\Omega_s$ and no more than n terms in the sum $\sum_{k=0}^{M-1} X_s(e^{j\omega + 2\pi k/M})$ be non-zero at any frequency.

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