DISCRETE TIME SIGNALS WHICH CAN BE RECOVERED FROM SAMPLES

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Abstract

If a discrete time signal $x(n)$ is bandlimited appropriately we can decimate it without aliasing. However, there exists a broad class of non bandlimited signals which can be recovered perfectly from their decimated versions. In this paper we consider both uniform and nonuniform decimation of this kind and explore some applications, especially in noise shaping and in Σ-Δ modulator type of architectures.

1. SAMPLING THEOREMS FROM SUBBAND CODERS

If a sequence $x(n)$ is bandlimited to $[-\pi/M, \pi/M]$ we can reconstruct it from its samples $x(Mn)$, commonly called the M-fold decimated version. More general kind of bandlimited sequences (e.g., bandpass, multiband, ...) can also be reconstructed from uniformly or nonuniformly sampled versions [1]. If a sequence is not bandlimited at all in any way, can we still do this? Sometimes yes, as the obvious example of Fig. 1.1 shows. Here $x(n)$ is the output of an interpolation filter $F(z)$ (all terms and notations are as in [1]). Thus $x(n) = \sum_k y(k) f(n-kM)$, and usually $F(z)$ is a Nyquist(M) filter [1], [3], [4], that is $f(Mn) = \delta(n)$. This means that $y(n)$ is itself equal to $x(Mn)$, and we have the relation $x(n) = \sum_k x(kM) f(n-kM)$. In other words, $x(n)$ is completely defined by the samples $x(Mn)$ even though it is not an ideally bandlimited signal [unless $F(e^{j\omega})$ is a ideal bandlimiter].

1.1. Multiband Models

More elaborate "sampling theorems" are hidden in filter bank structures as we now demonstrate. In Fig. 1.1 we considered a signal $x(n)$ which can be modeled as the output of an interpolation filter (e.g., $x(n)$ could be a dominantly lowpass signal). Consider an ℓ^2 signal $x(n)$ which can be modeled in the more general manner shown in Fig. 1.2(a), where $y_i(n)$ are ℓ^2 signals, $L \leq M$, and $F_i(z)$ are stable rational filters. This we call the *multiband model* because signals with multiband spectra (Fig. 1.2(b)) can be modeled or approximated like this. Suppose we express each filter $F_k(z)$ in its polyphase form $F_k(z) = \sum_{m=0}^{M-1} z^m R_{m,k}(z^M)$ [1], [2]. Then the signal model can be redrawn as in Fig. 1.3 where $\mathbf{R}_{M,L}(z)$ is $M \times L$. Clearly $x(n)$ is an interleaved version of the signals $x_i(n)$ indicated in the figure. In fact $x_i(n) = x(Mn-i)$, the ith polyphase component of $x(n)$. Suppose $\mathbf{R}(z)$ is an $L \times L$ submatrix of $\mathbf{R}_{M,L}(z)$, obtained by retaining the rows $i_0, i_1, \ldots i_{L-1}$. That is,

$$\begin{bmatrix} X_{i_0}(z) \\ X_{i_1}(z) \\ \vdots \\ X_{i_{L-1}}(z) \end{bmatrix} = \mathbf{R}(z) \begin{bmatrix} Y_0(z) \\ Y_1(z) \\ \vdots \\ Y_{L-1}(z) \end{bmatrix} \quad (1.1)$$

If [det $\mathbf{R}(z)$] is not zero anywhere on the unit circle, then $\mathbf{R}^{-1}(z)$ is stable. So we can reconstruct the L signals $y_i(n)$ from the L signals $x_{i_k}(n)$ in a stable (possibly noncausal) manner. This means we can recover all the M signals $x_i(n)$:

$$\begin{bmatrix} X_0(z) \\ X_1(z) \\ \vdots \\ X_{M-1}(z) \end{bmatrix} = \mathbf{R}_{M,L}(z)\mathbf{R}^{-1}(z) \begin{bmatrix} X_{i_0}(z) \\ X_{i_1}(z) \\ \vdots \\ X_{i_{L-1}}(z) \end{bmatrix} \quad (1.2)$$

Thus the original signal $X(z) = \sum_{n=0}^{M-1} z^n X_n(z^M)$ can be recovered from the set of L sampled versions

$$x_{i_k}(n) = x(Mn - i_k), \quad 0 \leq k \leq L-1. \quad (1.3)$$

Together, this set constitutes a nonuniformly sampled or decimated version of $x(n)$ as demonstrated in Fig. 1.4 for $M=5, L=3$. Summarizing, an ℓ^2 signal satisfying the model of Fig. 1.2(a) can be recovered from an (M/L)-fold nonuniformly decimated version.

The special case of Fig. 1.1 results when $L=1$. It can be verified that in this case the formula for reconstruction of $x(n)$ from $x(Mn-i)$ can be expressed as shown in Fig. 1.5 where the reconstruction filter $S(z) = F(z)/R_i(z^M)$. Recall $R_k(z)$ are the polyphase

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components, that is, $F(z) = \sum_{k=0}^{M-1} z^k R_k(z^M)$. If $F(z)$ is Nyquist(M), there would exist an i such that $R_i(z) = 1$ and we get $S(z) = F(z)$.

1.2. Stability of Reconstruction

The stability of the reconstruction scheme is addressed in a more elaborate manner in [5]. For example, in the simple model of Fig. 1.1 suppose $F(z)$ is not a Nyquist(M) filter. Then reconstruction from $x(Mn-i)$ (for some fixed i) is possible only if we define the reconstruction filter to be $S(z) = F(z)/R_i(z^M)$ where $R_i(z)$ is the ith polyphase component of $F(z)$. Stability of reconstruction requires that $R_i(z)$ be free from unit circle zeros. If there exists no polyphase component with this property, then $x(n)$ cannot in general be recovered in a stable manner from a uniform set of samples like $x(Mn-i)$. A subtle point here is that recovery from a nonuniformly decimated version could still be possible.

For example, it is shown in [5] that if the model filter $F(z) = 1 + z - z^2 + z^3$ then for $M = 2$, there is no stable way to recover $x(n)$ from the uniformly decimated versions $x(2n)$ or $x(2n-1)$. However, we can recover $x(n)$ from the signals $x(4n-2)$ and $x(4n-3)$ which together constitute a 2-fold nonuniformly decimated version. This reconstruction is not only stable, in fact it is FIR as shown in [5]. More generally, it is proved in [5] that if $F(z)$ is FIR and there exist two polyphase components $R_{k_0}(z)$ and $R_{k_1}(z)$ that are relatively prime, then stable reconstruction from $x(2Mn - i_0)$ and $x(2Mn - i_1)$ is possible for appropriate choice of constants i_0, i_1.

2. FINDING A SIGNAL MODEL

What kind of signals can be realistically modeled as in Fig. 1.1 or 1.2(a)? To answer this, recall the subband coder system, where a signal $x(n)$ is split into M bands, and reconstructed perfectly from maximally decimated versions (Fig. 2.1). Suppose $x(n)$ has most of its energy concentrated in L subbands, which we number as the first L subbands. Then the signal model Fig. 1.2(a) is a good approximation. Thus, given a signal $x(n)$ with energy concentrated mostly in certain subbands, the problem of finding the best signal model reduces to that of finding the filter bank that produces the most dominant L subbands.

Orthogonal projection interpretation. Suppose the filter bank is orthonormal (paraunitary) [1]. Then $x(n)$ has the orthonormal expansion

$$x(n) = \sum_{i=0}^{M-1} \sum_{k} y_i(k) f_i(n - kM). \qquad (2.1)$$

If the subbands $y_i(k), i \geq L$ are discarded as being not significant, then the result

$$x_P(n) = \sum_{i=0}^{L-1} \sum_{k} y_i(k) f_i(n - kM) \qquad (2.2)$$

is an orthogonal projection of $x(n)$ onto the subspace spanned by the first L filters only. Thus, we have projected a signal $x(n)$ into a subspace such that *the projection can be reconstructed from its samples.* Finding the best basis, that is, the best set of orthonormal filters $\{f_k(n)\}$ for a given signal (so that the error due to approximation is minimized) is the optimal modeling problem. This problem can be formulated and solved more quantitatively, but is outside the scope of our discussion here.

The Inverse of the Model

Consider the system of Fig. 2.2 where the signal $x(n)$ goes through an M-fold decimation filter $H(z)$. Assume $x(n)$ satisfies the model of Fig. 1.1. Suppose we choose $H(z)$ such that

$$H(z)F(z)\Big|_{\downarrow M} = 1, \qquad (2.3)$$

where the notation $A(z) = B(z)|_{\downarrow M}$ means $a(n) = b(Mn)$. The preceding equation therefore says that $H(z)F(z)$ is a Nyquist(M) filter. With this choice of $H(z)$ we claim that the output of Fig. 2.2 is $y(n)$, as indicated. That is, Fig. 2.2 acts as an inverse of Fig. 1.1. To prove this simply note that $X(z) = Y(z^M)F(z)$ from Fig. 1.1. Thus, the output of Fig. 2.2 is

$$Y(z^M)F(z)H(z)\Big|_{\downarrow M} = Y(z)\Big(F(z)H(z)\Big|_{\downarrow M}\Big) = Y(z)$$

indeed. Note that if $F(z)$ is one of the filters in an M channel orthonormal filter bank then we simply have $H(z) = \widetilde{F}(z)$ [1].

In a similar way, we can talk about an inverse of the system Fig. 1.2(a). Fig. 2.3 shows this inverse, which produces the signals $y_i(n)$ in response to the model signal $x(n)$. The filters $H_k(z)$ are related to $F_k(z)$ such that $H_k(z)F_m(z)|_{\downarrow M} = \delta(k-m)$. This resembles the biorthogonality condition satisfied by perfect reconstruction (PR) filter banks. Thus, we can imagine that the filters in Fig. 2.3 and 1.2(a) are a subset of L analysis and synthesis filters in an M channel maximally decimated PR filter bank.

Existence of model inverse. Given a rational transfer function $F(z)$, can we always construct $H(z)$ such that the product $H(z)F(z)$ is Nyquist(M)? Since we can trivially do this by letting $H(z) = 1/F(z)$, it is more interesting to put some constraints into the allowed solution. For example, suppose $F(z)$ is

FIR. Can we find FIR $H(z)$ such that $H(z)F(z)$ is Nyquist(M)? Consider the polyphase decomposition $F(z) = \sum_{i=0}^{M-1} z^i R_i(z^M)$. If the polynomials $\{R_i(z)\}$ do not have a common factor $C(z)$, then we can indeed find such an FIR $H(z)$. This is because, by an extension of Euclid's theorem there exist polynomials (FIR filters) $\{E_i(z)\}$ such that $\sum_i E_i(z)R_i(z) = 1$. Defining $H(z) = \sum_i z^{-i}E_i(z^M)$ we then verify that $H(z)F(z)$ is a Nyquist(M) filter indeed. If $R_i(z)$ do have a common factor $C(z)$ of order ≥ 1, then $F(z)$ has the factor $C(z^M)$, that is, $F(z)H(z) = C(z^M)F_1(z)H(z)$ which cannot be Nyquist(M) for FIR $H(z)$.

3. APPLICATIONS

If a signal $x(n)$ originates at a place where sophisticated signal processing is not practical, and if we know that it can be reasonably approximated by a model like Fig. 1.1 (more generally Fig. 1.2(a)), then we can perform data compression by direct decimation (nonuniform for the multiband model). The decimated version can then be recovered at a "receiver end" by means of linear filtering.

Another simple application arises when we quantize the signal. Suppose there is a certain constraint on the bit rate so that a direct quantization of $x(n)$ can use only b bits per sample. If we transmit the decimated version $x(Mn)$, then in principle we can use Mb bits per sample and the reconstructed version of $x(n)$ is much more accurate that a b-bit version of $x(n)$. (An exact analysis can be done [**5**]). Thus, if we know that $x(n)$ satisfies the interpolator model, we can exploit it for efficient use of bits.

Noise Shaping

What else can we do if we are aware that $x(n)$ can be reconstructed from its samples (even though not bandlimited)? Let us take a line through the kind of things we would do if it were *exactly* bandlimited (that is, an oversampled signal). In that case, we can do noise shaping (as in Σ-Δ modulation) and quantize it to very few bits, perhaps one bit. Can we do a similar thing with a non bandlimited signal satisfying the model of Fig. 1.1, more generally Fig. 1.2(a)?

Σ-Δ modulators. We show how to do this for the simple case of Fig. 1.1. First assume that $x(n)$ is an oversampled lowpass signal, and recall how the Σ-Δ modulator operates (Fig. 3.1): The prefilter $P(z)$ would typically (though not necessarily) be a "lossy integrator", that is, $P(z) = 1/(1-\alpha z^{-1})$ with $0 < \alpha < 1$. The quantizer Q would typically be a delta modulator at the transmitter followed by a demodulator at the receiver. Since $1/P(z)$ is highpass, the quantization noise component gets shaped so that most noise energy moves to the high frequency region. The postfilter $T(z)$ is lowpass, and therefore attenuates the noise significantly. The "signal component" at the output is precisely equal to $x(n)$ if $x(n)$ is strictly bandlimited to π/M and if $T(z)$ is ideal lowpass.

For the case where $x(n)$ is not really bandlimited but satisfies the model of Fig. 1.1 for some "reasonable" lowpass filter $F(z)$, how do we mimic the noise shaping idea of the Σ-Δ modulator? Fig. 3.2 shows one possibility. Instead of the lowpass filter $T(z)$ we have a more elaborate multirate scheme which involves two filters $H(z)$ and $F(z)$. Here $F(z)$ is just the model filter (Fig. 1.1), and $H(z)$ is the filter appearing in the model inverse (Fig. 2.2). By the model inverse property we see that the reconstructed signal $\widehat{x}(n) = x(n)$ in absence of the quantizer Q. This is the motivation for replacing $T(z)$ in the traditional Σ-Δ modulator with the multirate cascade in Fig. 3.2. If $H(z)$ is a good lowpass approximation, the noise shaping idea works. At least for the case where $\widetilde{F}(z)F(z)$ is Nyquist(M) [so that $H(z) = \widetilde{F}(z)$], we know that this is the case.

The preceding structure opens up interesting research problems. For example, for fixed $F(z)$ (and $H(z)$), what is the best prefilter $P(z)$ that minimizes the output reconstruction error (e.g., in the m.s. sense)? What is the best stable rational (better still, FIR) $P(z)$ of fixed order with stable inverse that minimizes the reconstruction error? One could address this meaningfully by imposing a statistical model in Fig. 3.2, and defining an appropriate mean square error measure. The problem can be regarded as an extension, to the multirate case, of the half-whitening problem [**6**]. Details can be found in [**5**].

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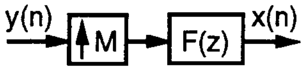

Fig. 1.1. A signal model.

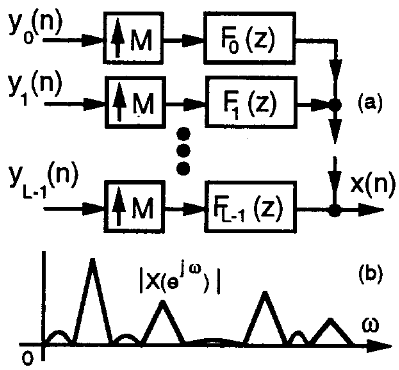

Fig. 1.2 (a) The multiband signal model, and (b) typical signal spectrum.

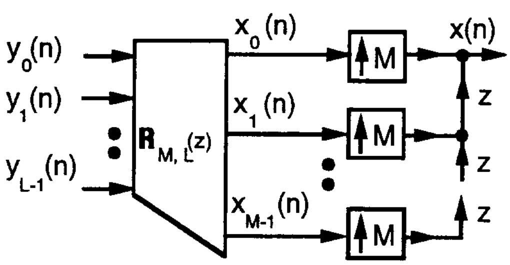

Fig. 1.3. Polyphase form of Fig. 1.2(a).

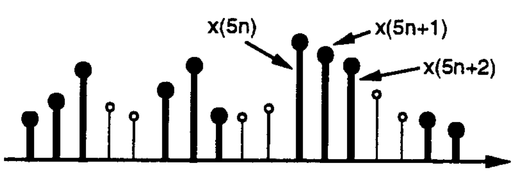

Fig. 1.4. Nonuniform decimation with M=5,L=3.

Fig. 1.5. The reconstruction scheme.

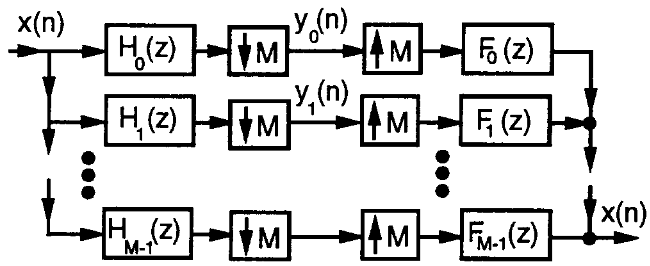

Fig. 2.1. The M channel subband coder.

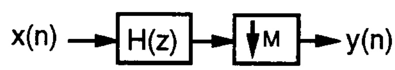

Fig. 2.2. The model inverse.

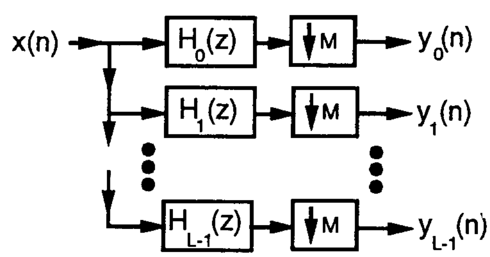

Fig. 2.3. The multiband model inverse.

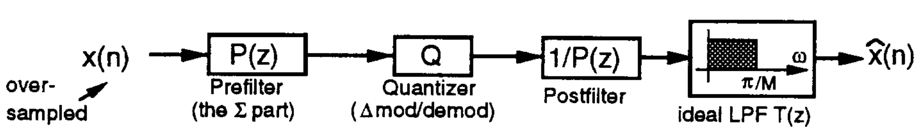

Fig. 3.1. Schematic of traditional sigma-delta modulator.

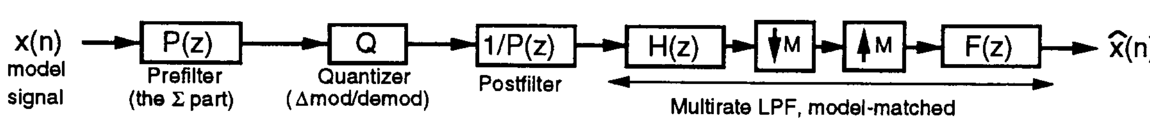

Fig. 3.2. Schematic of new sigma-delta modulator.