ANALYSIS OF THE QR-RLS ALGORITHM FOR COLORED-INPUT SIGNALS

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ABSTRACT

A detailed analysis of the QR-RLS algorithm in finite and infinite precision implementations is presented, emphasizing the case where the input signal samples are correlated. The expressions for the mean square values of all internal variables in steady state are first derived. These expressions are key to determine the dynamic range of the internal signals, and to derive the analytical expressions for the mean square values of the deviations in the output variables of the algorithm in finite wordlength implementations. Previous works address this problem considering the input signal a white noise, a situation not so often encountered in practice. The accuracy of all analytical results are verified through a number of computer simulations.

1. INTRODUCTION

The recursive least squares (RLS) algorithms for FIR adaptive filtering are particularly attractive due to their fast convergence, specially for correlated input signals. The RLS algorithm based on the QR decomposition (QR-RLS) is one of the most attractive due to its numerical stability [1]-[4], and systolic array implementation [5].

For uncorrelated input signals the mean square values of the internal variables of the Conventional QR-RLS algorithm was derived in [2], [6]. A set of relations, for calculating the mean square values of deviations in the adaptive filter outputs when implemented with fixed-point arithmetic was recently presented in [4].

This paper extends these results for correlated input signals. As will be shown the results for correlated inputs are quite different from the results of [2], [4] for uncorrelated input signals. The accuracy of the derived relations are verified through some computer simulations.

2. QR-RLS ALGORITHM

The algorithm analysed in this paper is the conventional QR-RLS algorithm described as follows:

Matricial Formulation: For $i = 0, \cdots, N$, do

$$\cos\theta_i(k) = \frac{\lambda^{1/2}u_{i,i}(k-1)}{\sqrt{\lambda u_{i,i}^2(k-1)+a_{i,i}^2(k)}} \quad (1)$$

$$\sin\theta_i(k) = \frac{a_{i,i}(k)}{\sqrt{\lambda u_{i,i}^2(k-1)+a_{i,i}^2(k)}} \quad (2)$$

$$Q'_i(k) = \begin{bmatrix} \cos\theta_i(k) & & -\sin\theta_i(k) & \\ & I_{k-N+i-1} & & \\ \sin\theta_i(k) & & \cos\theta_i(k) & \\ & & & I_{N-i} \end{bmatrix}$$

$$X^p(k) = \begin{bmatrix} x^H(k) \\ 0_{k-N-1,N+1} \\ \lambda^{1/2}U(k-1) \end{bmatrix} \quad (3)$$

$$\hat{X}(k) = Q'_N(k)\cdots Q'_0(k)X^p(k) = \bar{Q}(k)X^p(k) \quad (4)$$

$$\hat{X}(k) = \begin{bmatrix} 0_{k-N,N+1} \\ U(k) \end{bmatrix} \quad (5)$$

$$d^p(k) = \begin{bmatrix} d^*(k) \\ \lambda^{1/2}\hat{d}_1(k-1) \\ \lambda^{1/2}\hat{d}_2(k-1) \end{bmatrix} \quad (6)$$

$$\hat{d}(k) = Q'_N(k)\cdots Q'_0(k)d^p(k) = \bar{Q}(k)d^p(k) = \begin{bmatrix} e_q(k) \\ \hat{d}_1(k) \\ \hat{d}_2(k) \end{bmatrix} \quad (7)$$

Error calculation:

$$e(k) = e_q(k)\cos\theta_N(k)\cdots\cos\theta_0(k) \quad (8)$$

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where $\boldsymbol{x}(k)$ is the input signal vector, $d(k)$ is the reference signal, and $e(k)$ is the error signal.

The quantities $a_{i,j}(k)$ and $b_i(k)$ not explicitly shown, represent intermediate values that appear in equations (4) and (7) respectively.

3. SOME USEFUL RELATIONS

In the present section, a number of known relations, that are also valid for the analysis of the QR-RLS algorithm with colored input signals, are listed for convenience. The relations are:

$$E\{a_{0,j}^2(k)\} = \sigma_x^2 \tag{9}$$

for $j = 0, 1, \cdots, N$.

$$E\{a_{i,i}^2(k)\} = (1-\lambda)E\{u_{i,i}^2(k)\} \tag{10}$$

$$\frac{E\{u_{i,i}^2(k-1)\}}{E\{u_{i,i}^2(k)\}} \approx 1 \tag{11}$$

$$\frac{E\{a_{i,i}^2(k)\}}{E\{u_{i,i}^2(k)\}} \approx 1-\lambda \tag{12}$$

An important relation is the definition of the intermediate variable $a_{i,j}(k)$ given by

$$\begin{aligned} a_{i,j}(k) &= -\lambda^{\frac{1}{2}} u_{i-1,j}(k-1)\frac{a_{i-1,i-1}(k)}{u_{i-1,i-1}(k)} \\ &+ \lambda^{\frac{1}{2}} a_{i-1,j}(k)\frac{u_{i-1,i-1}(k-1)}{u_{i-1,i-1}(k)} \end{aligned} \tag{13}$$

Another recurrence that is frequently computed in the QR-RLS algorithm is

$$u_{i,j}(k) = -\lambda u_{i,j}(k-1)\frac{u_{i,i}(k-1)}{u_{i,i}(k)} + a_{i,j}(k)\frac{a_{i,j}(k)}{u_{i,i}(k)} \tag{14}$$

where $u_{i,j}(k)$ are the elements of the matrix $\boldsymbol{U}(k)$.

4. ANALYSIS FOR COLORED INPUTS

In this section are derived the mean square value of several quantities related to the QR-RLS algorithm, namely $u_{i,j}(k)$, $a_{i,j}(k)$, $\hat{d}_{2,i}(k)$, $b_i(k)$ and $e_q(k)$. The analysis is based on the assumption that the input signal is a non-white stationary random signal where $r(j) = E\{x(i)x(i-j)\}$. The results presented here generalizes the ones presented for uncorrelated inputs.

A first assumption assumed throughout, that was confirmed in the simulation results, is that the mean square value of $\cos\theta_i(k)$, $\sin\theta_i(k)$ are given by (11) and (12), respectively. Also the square values of the cosines and sines were considered independent of the remaining quantities of the algorithm [6]. Using these assumptions and from (14), one can easily show that

$$E\{u_{i,j}(k)u_{i,i}(k)\} = \frac{E\{a_{i,j}(k)a_{i,i}(k)\}}{1-\lambda} \tag{15}$$

for $i \neq j$.

After some tedious calculations we concluded that $E\{u_{i,j}(k)a_{i,l}(k)\}$ is small as compared with $E\{u_{i,j}(k)u_{i,i}(k)\}$ and $E\{a_{i,j}(k)a_{i,i}(k)\}$, for any i, j, and l. Also assuming that $u_{i,j}(k)u_{i,i}(k)$ and $a_{i,j}(k)a_{i,i}(k)$ are statistically independent, and employing the result of (15), it can be shown that

$$\begin{aligned} E\{u_{i,j}^2(k)\} &\approx \frac{1}{1+\lambda}E\{a_{i,j}^2(k)\} \\ &+ \frac{2\lambda}{(1-\lambda^2)}\frac{E^2\{a_{i,j}(k)a_{i,i}(k)\}}{E\{a_{i,i}^2(k)\}} \end{aligned} \tag{16}$$

The mean square value of the intermediate variable $a_{i,j}(k)$ can be derived using the assumptions discussed so far, with the result being as follows

$$\begin{aligned} E\{a_{i,j}^2(k)\} &\approx \lambda(1-\lambda)E\{u_{i-1,j}^2(k-1)\} \\ &+ \lambda E\{a_{i-1,j}^2(k)\} \\ &- 2\lambda E\{a_{i-1,j}(k)a_{i-1,i-1}(k)\} \\ &\quad \frac{E\{u_{i-1,j}(k-1)u_{i-1,i-1}(k-1)\}}{E\{u_{i-1,i-1}^2(k)\}} \end{aligned} \tag{17}$$

In order to obtain a closed form solution for the equations (16) and (17) above, the following result is required

$$E\{a_{i,j}(k)a_{i,i}(k)\} \approx 2\lambda(1-\kappa)E\{a_{i-1,i}(k)a_{i-1,j}(k)\} \tag{18}$$

where σ_x^2 is the variance of the input signal and $\kappa = \frac{r(1)}{\sigma_x^2}$. The equation above can be solved iteratively using the following relation as starting point

$$E\{a_{1,j}(k)a_{1,1}(k)\} \approx 2\lambda(1-\frac{r(1)}{\sigma_x^2})r(j-i) \tag{19}$$

With the results described above, we can derive the mean square values of the internal quantities for correlated inputs, by following the calculation procedure described below

Initialization

$$E\{u_{0,0}^2(k)\} = \frac{\sigma_x^2}{1-\lambda} \tag{20}$$

For $j = 0, ..., N$, do

$$E\{a_{0,j}^2(k)\} = \sigma_x^2 \tag{21}$$

For $j = 1, ..., N$, do

$$\begin{aligned} E\{u_{0,j}^2(k)\} &\approx \frac{1}{1+\lambda} E\{a_{0,j}^2(k)\} \\ &+ \frac{2\lambda r^2(j)}{(1-\lambda^2)\sigma_x^2} \end{aligned} \tag{22}$$

For $i = 1, ..., N$, do
For $j = i, ..., N$, do

$$\begin{aligned} & E\{a_{i,j}^2(k)\} \approx \\ & \lambda(1-\lambda)E\{u_{i-1,j}^2(k-1)\} \\ + \; & \lambda E\{a_{i-1,j}^2(k)\} \\ - \; & \frac{(2\lambda)^{2i-1}(1-\kappa)^{2(i-1)}r^2(j-i+1)}{E\{a_{i-1,i-1}^2(k)\}} \end{aligned} \tag{23}$$

for $j = i$, do

$$E\{u_{i,i}^2(k)\} = \frac{E\{a_{i,i}^2(k)\}}{1-\lambda} \tag{24}$$

else

$$\begin{aligned} E\{u_{i,j}^2(k)\} &\approx \frac{1}{1+\lambda} E\{a_{i,j}^2(k)\} \\ &+ \frac{(2\lambda)^{2i+1}(1-\kappa)^{2i}r^2(j-i)}{(1-\lambda^2)E\{a_{i,i}^2(k)\}} \end{aligned} \tag{25}$$

The expressions of $E\{\hat{d}_{2,i}^2(k)\}$ and $E\{b_{i+1}^2(k)\}$ for uncorrelated inputs, given respectively by equations (26) and (27), remain valid for correlated inputs. A key result to show it, is the fact that $E\{u_{i,j}(k)u_{i,l}(k)\}$, for $j \neq l$ is small as compared with $E\{u_{i,j}^2(k)\}$. The proofs are omitted for the sake of brevity.

$$E\{\hat{d}_{2,i}^2(k)\} \approx \sum_{j=i}^{N} E\{u_{i,j}^2(k)\}E\{w_j^2(k)\} \tag{26}$$

$$\begin{aligned} E\{b_{i+1}^2(k)\} &\approx \sum_{j=1}^{i+1} \lambda^{i-j+2}(1-\lambda)E\{\hat{d}_{2,j}^2(k)\} \\ &+ \lambda^{i+1}E\{d^2(k)\} \end{aligned} \tag{27}$$

Simulations were performed to show the accuracy of the proposed formulas. A colored input sequence $x(k)$ was generated by the following operation $x(k) = g(k) * h(k) + v(k)$, where $h(k)$ is a sequence defined by $h(k) = \{0.5 + 0.5\cos(2\pi/W), 1, 0.5 + 0.5\cos(2\pi/W)\}$, $g(k)$ is a polar sequence with $g(k) = \pm 0.01$ and $v(k)$ is white gaussian noise with zero mean and variance $\sigma_v^2 = -30$ dB. By choosing the appropriate values for W, it is possible to control the eigenvalue spread $\mathcal{X}(\boldsymbol{R})$ of the input sequence $x(k)$ [1]. Some of the obtained values for the mean square values of $u_{i,j}(k)$ are shown on tables 1 and 2. The simulations used $\lambda = 0.99$, 5000 samples and an average over 5 experiments.

5. MEAN SQUARED VALUES OF DEVIATIONS IN THE INTERNAL VARIABLES

In order to generate analytical expressions for the excess of of mean square error, and for the variance of the deviation in the tap coefficients of the adaptive filter due to quantization, it is necessary to analyse the propagation of the quantization errors and to derive the mean square values of the deviations in all internal variables. Both tasks were previouly performed for the QR-RLS algorithm with white input signal [4], [7]. The results presented in [4] and [7] are also approximately valid for non-white input signals, if the same simplifying assumptions discussed in [7] are made. Therefore, by applying the results of the infinite precision analysis for non-white inputs to the expressions obtained through the propagation analysis, we obtain good estimates for the mean square errors in the internal and external variables of the QR-RLS algorithm. In order to verify these results, an identification problem was simulated using as input signal the colored noise described in the simulations of the previous section. The quantiy of interest is the excess of mean square error caused by finite precision implementation. The results obtained from simulations and from the proposed analysis are shown in tables 3 and 4. As can be seen, the expressions proposed are in close agreement with the simulation results.

6. CONCLUDING REMARKS

In this paper, we have presented a set of relations that predicts the mean square value of the internal variables of the conventional QR-RLS algorithm, when the input signal is a colored noise. With the new presented results, the previous solutions proposed for white noise inputs fall within this framework.

The proposed relations are key to predict the finite wordlength effects in the actual algorithm implementation, and to calculate the dynamic range of each internal variable of the algorithm. From another point of view, these relations allow the designer of an application specific hardware to determine the required wordlength such that the algorithm meets prescribed specifications, such as limited excess of mean square error due to quantization effects.

The expressions proposed for the mean square values of the internal variables were shown to be very accurate through simulation results.

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Table 1: $E\{u_{i,j}^2(k)\}$ with colored noise input.

$E\{u_{i,j}^2(k)\}$	$\mathcal{X}(R) = 6.08$		$\mathcal{X}(R) = 11.12$	
	Sim. (dB)	Calc. (dB)	Sim. (dB)	Calc. (dB)
$E\{u_{0,0}^2(k)\}$	-19.3	-19.6	-19.1	-19.4
$E\{u_{0,1}^2(k)\}$	-28.1	-27.4	-26.0	-25.6
$E\{u_{0,2}^2(k)\}$	-40.0	-41.2	-38.8	-39.5
$E\{u_{0,3}^2(k)\}$	-41.4	-42.6	-41.2	-42.4
$E\{u_{1,1}^2(k)\}$	-19.9	-18.8	-20.0	-18.2
$E\{u_{1,2}^2(k)\}$	-27.9	-26.7	-25.7	-24.4
$E\{u_{1,3}^2(k)\}$	-40.8	-41.0	-38.8	-39.0

Table 2: $E\{u_{i,j}^2(k)\}$ with colored noise input.

$E\{u_{i,j}^2(k)\}$	$\mathcal{X}(R) = 27.71$		$\mathcal{X}(R) = 46.82$	
	Sim. (dB)	Calc. (dB)	Sim. (dB)	Calc. (dB)
$E\{u_{0,0}^2(k)\}$	-18.8	-19.1	-18.5	-18.9
$E\{u_{0,1}^2(k)\}$	-24.7	-24.2	-23.5	-23.3
$E\{u_{0,2}^2(k)\}$	-37.2	-37.8	-35.0	-36.2
$E\{u_{0,3}^2(k)\}$	-40.0	-42.1	-39.0	-41.8
$E\{u_{1,1}^2(k)\}$	-20.1	-17.3	-20.2	-16.9
$E\{u_{1,2}^2(k)\}$	-24.2	-22.7	-23.0	-21.3
$E\{u_{1,3}^2(k)\}$	-36.9	-36.8	-34.5	-34.7

Table 3: $E\{[\Delta e(k)]^2\}$ with colored noise input (24 bits).

$\mathcal{X}(\boldsymbol{R})$	Sim. (dB)	Calc. (dB)
6.08	-137.7	-138.2
11.12	-136.2	-136.8
27.71	-135.9	-136.7
46.82	-135.5	-136.7

Table 4: $E\{[\Delta e(k)]^2\}$ with colored noise input (15 bits).

$\mathcal{X}(\boldsymbol{R})$	Sim. (dB)	Calc. (dB)
6.08	-83.1	-84.0
11.12	-82.2	-83.1
27.71	-81.8	-82.7
46.82	-80.9	-81.7