A STEADY-STATE ANALYSIS OF THE LMS ADAPTIVE ALGORITHM WITHOUT USE OF THE INDEPENDENCE ASSUMPTION

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ABSTRACT

Current analysis of the LMS algorithm makes use of an "independence assumption" stating statistical independence of successive input vectors. This assumption conflicts with the inherent deterministic coherence of the vector input signal and, as such, is the source of conceptual and didactic difficulties. Nevertheless, due to its analytic convenience and its moderate agreement with experimental results, it is in widespread use. In this paper, a theory of the steady-state behaviour of the LMS algorithm is presented that avoids the independence assumption with its inherent problems and yields a number of new results. Simulations support the analytic conclusions.

1. THE WEIGHT-ERROR CORRELATION MATRIX (WECM) AND THE INDEPENDENCE ASSUMPTION

An adaptive filter is designed to imitate some actual or hidden reference filter such that it gradually moves from an arbitrary unadapted state in the direction of the reference filter. This "initial learning" goes on far away from the adapted state. For the common algorithms such as the LMS algorithm studied in this paper the final stage of adaptation exhibits a different pattern. Then the theoretically determined tuning of the adaptive filter known as the "Wiener solution" is not ultimately reached as a limiting value, but turns out to be superimposed by persistent random fluctuations. Their amplitude is coupled with the adaptation rate in the sense that a fast adaptation involves large final parameter fluctuations of the adaptive filter. The free constant controlling the combination adaptation speed + parameter fluctuations is called the adaptation constant μ.

In this paper we study the parameter or weight fluctuations for the well-known LMS algorithm applied to the common FIR filter structure. Under steady-state conditions (constant reference filter, stationary excitation of the adaptive filter) the second-order statistics of the weight fluctuations are characterized by a correlation matrix. Its diagonal terms represent the squared amplitudes of the pertinent fluctuations, whereas the off-diagonal entries stand for the correlations between pairs of weight fluctuations. Since we are dealing with the deviations from the Wiener solution, the correlation matrix is usually referred to as the "weight error correlation matrix" WECM, denoted by V.

In the current literature V is throughout determined with the aid of an "independence assumption" stating statistical independence of successive input vectors. Such an assumption can be well argued for a "true" vector signal like that emerging from a sensor array. In the case of an FIR filter consisting of a tapped delay-line fed by a scalar input signal the input vector is made up of successive values of the scalar signal and, as such, has a more artificial nature. Within an updating cycle the various vector components are shifted to the next place with the last component removed and the first component renewed. Thus, there is a strong deterministic coherence between successive input vectors clearly conflicting with the independence assumption. Nevertheless the assumption is in common use, particularly because of the provided analytical convenience and the reasonable agreement with measured results [1-5]. To be sure, recent work [6,7] avoids the independence assumption but it has not yet yielded significant analytical results.

In this paper we develop an iterative approach without the independence assumption finally leading to a power series of the WECM in terms of the adaptation constant. However, only the first few terms of the series are actually determined so that the present analysis should be indicated as a "small-signal theory", valid for small adaptation constants. There are no basic restrictions with respect to the spectral distributions or the probability distributions of the exciting signals, but special assumptions (whiteness, narrowband, Gaussian) can considerably simplify the results.

A scalar stationary, zero-mean signal x_k is assumed to simultaneously excite a fixed FIR-filter with weight vector $\underline{h}$ and the adaptive filter under consideration with the time-varying weight vector $\underline{w}_k$. For convenience, both vectors have the same dimension M. The output of the reference filter is contaminated by an additive noise n_k, statistically independent of x_k and, like x_k, stationary and with zero mean. By taking the difference between the sum signal and the output signal y_k of the adaptive filter we form the error signal e_k, which controls the adaptation algorithm.

For further use we define the $M \times 1$ input vector as $\underline{x}_k = (x_k, x_{k-1}, \cdots, x_{k-M+1})^t$, so that the outputs of the reference and the adaptive filter can be written as $\underline{h}^t\underline{x}_k$ and $\underline{w}_k^t\underline{x}_k$, respectively. Further, the weight error vector is defined as $\underline{v}_k = \underline{w}_k - \underline{h}$ so that the error signal becomes

$$e_k = n_k + \underline{h}^t\underline{x}_k - y_k = n_k - \underline{v}_k^t\underline{x}_k.$$

The LMS algorithm is given by

$$\underline{w}_{k+1} = \underline{w}_k + 2\mu e_k \underline{x}_k$$

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so that the weight error satisfies the state equation

$$\underline{v}_{k+1} = \underline{v}_k - 2\mu \underline{x}_k \underline{x}_k^t \underline{v}_k + 2\mu n_k \underline{x}_k, \tag{1}$$

which is the starting point for the determination of the WECM

$$V = E\{\underline{v}_k \underline{v}_k^t\}. \tag{2}$$

2. NORMALIZATION AND GENERALIZATION OF THE STATE EQUATION AND ITS SOLUTION BY ITERATION

Next we absorb the adaptation constant into the pertinent signals through introducing normalized signals $\sqrt{2\mu}n_k$ and $\sqrt{2\mu}\underline{x}_k$ which, for sake of simplicity, are again denoted by n_k and $\underline{x}_k$, respectively. This way (1) passes into

$$\underline{v}_{k+1} = \underline{v}_k - \underline{x}_k \underline{x}_k^t \underline{v}_k + n_k \underline{x}_k. \tag{3}$$

Notice that we have not actually lost a free control parameter. The former freedom of choice for μ is now replaced with the freedom to choose the power of the input signal[1]

$$P = E\{x_k^2\}. \tag{4}$$

(In this sense the need for the adaptation constant can be questioned at all, unless the input signal in (1) is normalized in amplitude). State equation (3) is a representative of a wider class of the type

$$\underline{v}_{k+1} = \underline{v}_k - R_k \underline{v}_k + \underline{f}_k, \tag{5}$$

where R_k is a symmetric, positive (semi-)definite time-varying $M \times M$ matrix and $\underline{f}_k$ is an $(M \times 1)$ excitation vector. These quantities are uncorrelated, stationary random signals with $E\{\underline{f}_k\} = \underline{0}$. Indeed, in our special case R_k becomes the positive semi-definite dyadic product $\underline{x}_k \underline{x}_k^t$, while $E\{\underline{f}_k\} = E\{n_k\}E\{\underline{x}_k\} = \underline{0}$ holds with all entries of R_k uncorrelated with those of $\underline{f}_k$, due to the statistical independence of n_k and x_k. Contrary to $\underline{f}_k$, the matrix R_k does not have zero mean. The mean value $R = E\{R_k\}$ (called "correlation matrix" in our case) is throughout positive definite (only in exceptional cases it is semi-definite). The local (semi-)positiveness of R_k admits an interpretation of the term $(-R_k \underline{v}_k)$ in (5) as a time-varying system damping. Without that term (5) would represent an ideal integrator (in continuous-time terminology), which has to be reckoned among the unstable systems.

For $R_k \neq 0$ and thus $R \neq 0$ the instability is removed in the sense that the solution $\underline{v}_k$ of (5) no longer contains an infinite zero-frequency component. For a small R, the weight-error $\underline{v}_k$ contains strong low-frequency components so that $\underline{v}_k$ is a slowly varying signal with weak high-frequency components superimposed. This statement leads to the fundamental proposition that for $R \to 0$ the time-varying R_k in (5) can be replaced with its constant mean value R without noticeably affecting the solution $\underline{v}_k$. In other words, the solution $\underline{\alpha}_k$ of the modified state equation

$$\underline{\alpha}_{k+1} = \underline{\alpha}_k - R\underline{\alpha}_k + \underline{f}_k \tag{6}$$

is an approximation of $\underline{v}_k$ such that $\underline{v}_k \to \underline{\alpha}_k$ for $R \to 0$. The truth of this statement can be supported by the following heuristic reasoning. For $P \to 0$ implying $R \to 0$ the $\underline{v}_k$ fluctuations become so slow that in the product $R_k \underline{v}_k$ in (5) the second factor may be considered constant in some finite discrete time interval of length n: $\underline{v}_k \approx \underline{v}$. Adding the equations (5) pertaining to that interval then leads to a large-scale updating mechanism. The crucial term $\sum R_k \underline{v}_k$ can then be replaced with $(\sum R_k)\underline{v} \approx nR\underline{v}$, where, due to ergodicity, the time average has been replaced with an ensemble average. For this to be true the length n of the time interval has, of course, to be sufficiently great, which is permitted if R is sufficiently small. The "proof" is completed with the observation that the expression $nR\underline{v}$ is also obtained if for all k the time-varying R_k is replaced with its average value R, as is done in (6).

Thus for $R \to 0$ we can solve the simpler difference equation (6) with constant coefficients rather than the time-varying problem (5). The statement "R is small" amounts to "P is small" or (in the original terminology) "μ is small"; throughout conditions under which $\underline{v}_k$ can be approximated by $\underline{\alpha}_k$. An attractive concomitant feature of this approximation is that $\underline{\alpha}_k$ can be used as starting point for an iteration generating a whole set of solutions of (5) whose accuracy increases with increasing iteration order, or, stated otherwise, whose domain of validity increases, viewed as a function of R.

The iteration yields $\underline{v}_k$ in (5) as a sum of partial functions

$$\underline{v}_k = \underline{\alpha}_k + \underline{\beta}_k + \underline{\gamma}_k + \cdots, \tag{7}$$

of which the previously found $\underline{\alpha}_k$ is the leading term. After introduction of the zero-mean matrix $P_k = R_k - R$, insertion of (7) into (5) yields

$$\begin{aligned} &\underline{\alpha}_{k+1} + \underline{\beta}_{k+1} + \underline{\gamma}_{k+1} + \cdots \\ &= (I - R)(\underline{\alpha}_k + \underline{\beta}_k + \underline{\gamma}_k + \cdots) \\ &\quad - P_k(\underline{\alpha}_k + \underline{\beta}_k + \cdots) + \underline{f}_k. \end{aligned} \tag{8}$$

Appropriate assembling of equal-order terms yields the iteration set

$$\left.\begin{aligned} \underline{\alpha}_{k+1} &= (I - R)\underline{\alpha}_k + \underline{f}_k \\ \underline{\beta}_{k+1} &= (I - R)\underline{\beta}_k - P_k\underline{\alpha}_k \\ \underline{\gamma}_{k+1} &= (I - R)\underline{\gamma}_k - P_k\underline{\beta}_k \\ &\vdots \end{aligned}\right\} \tag{9}$$

The first equation (equal to (6)) determines $\underline{\alpha}_k$ from $\underline{f}_k$, after that $\underline{\beta}_k$ follows from $\underline{\alpha}_k$, $\underline{\gamma}_k$ follows from $\underline{\beta}_k$ etc. Thus we have the chain $\underline{f}_k \to \underline{\alpha}_k \to \underline{\beta}_k \to \underline{\gamma}_k \cdots$, in which for sufficiently small R the individual terms decay to such a degree that the series (7) converges and thus can be truncated after a sufficient number of terms. Observe that all difference equations (9) belong to the same type, so that the same (linear, shift-invariant) operator applies to the various signal transformations:

$$\left.\begin{aligned} \underline{\alpha}_k &= \mathcal{L}\{\underline{f}_k\} \\ \underline{\beta}_k &= \mathcal{L}\{-P_k\underline{\alpha}_k\} \\ \underline{\gamma}_k &= \mathcal{L}\{-P_k\underline{\beta}_k\} \\ &\vdots \end{aligned}\right\} \tag{10}$$

[1] The freedom to choose the power of n_k is of no special interest, due to the linear dependence of $\underline{v}_k$ on n_k.

This operator can be explicitly written in the form

$$\begin{aligned} \underline{\alpha}_k &= \mathcal{L}\{\underline{f}_k\} = \sum_{j=-\infty}^{\infty} H_j \underline{f}_{k-j}, \\ H_j &= \varepsilon_{j-1}(I-R)^{j-1}, \\ \varepsilon_j &= \begin{cases} 0 & \text{for} \quad j<0, \\ 1 & \text{for} \quad j \geq 0. \end{cases} \end{aligned} \tag{11}$$

Notice that the matrix impulse response H_j starts at $j=1$, so that only past values of $\underline{f}_k$ contribute to $\mathcal{L}\{\underline{f}_k\}$.

The recursion mechanism transforms the time-varying parameter R_k in (5) into a set of excitation functions $(-P_k\underline{\alpha}_k)$, $(-P_k\underline{\beta}_k)$, etc. serving as source terms in simple constant-coefficient state equations. The latter represent an operator $\mathcal{L}\{.\}$ with low-pass character whose cut-off frequency is extremely low for $R \to 0$ corresponding to very slow variations of the operator's output signal.

Without going into details (see [8]) we maintain that in the chain $\underline{f}_k \to \underline{\alpha}_k \to \underline{\beta}_k \to \underline{\gamma}_k \cdots$ the individual recursion terms decrease such that the ratio of subsequent signal powers is in the order of P (which is proportional to R). With $\underline{f}_k = n_k\underline{x}_k$ and $R_k = \underline{x}_k\underline{x}_k^t$ we then have $E\{n_k^2\} = \mathcal{O}(1)$, $E\{\underline{f}_k^t\underline{f}_k\} = \mathcal{O}(P)$, $E\{\underline{\alpha}_k^t\underline{\alpha}_k\} = \mathcal{O}(1)$, $E\{\underline{\beta}_k^t\underline{\beta}_k\} = \mathcal{O}(P)$, $E\{\underline{\gamma}_k^t\underline{\gamma}_k\} = \mathcal{O}(P^2)$, etc. (The first statement comprehends a normalization of all powers with respect to that of the reference signal). Clearly the operator $\mathcal{L}\{.\}$ has an amplifying property with a power amplification factor proportional to P^{-1}. According to (10), this amplification is overcompensated by the multiplicative factors $(-P_k)$ contributing an attenuation proportional to P^2, which ultimately results in a net attenuation per iteration cycle proportional to P. This way the convergence of the iteration is guaranteed for sufficiently small P values.

3. SERIES EXPANSION OF THE WECM

With (7) we can write the required WECM in the form

$$V = E\{\underline{v}_k\underline{v}_k^t\} = E\{(\underline{\alpha}_k + \underline{\beta}_k + \underline{\gamma}_k + \cdots)(\underline{\alpha}_k^t + \underline{\beta}_k^t + \underline{\gamma}_k^t + \cdots)\} \tag{12}$$

which, in elaborated form, equals a sum of partial correlations between the various signals $\underline{\alpha}_k$, $\underline{\beta}_k$, $\underline{\gamma}_k$, $\cdots$. Each of these partial correlations is an $\mathcal{O}(P^\nu)$ term, where the lowest exponent $\nu = 0$ occurs for the combination $E\{\underline{\alpha}_k\underline{\alpha}_k^t\}$, while ν increases with increasing iteration order of both signals involved. Without proof (cf. [8]) we have $E\{\underline{\alpha}_k\underline{\alpha}_k^t\} = \mathcal{O}(1)$, $E\{\underline{\beta}_k\underline{\alpha}_k^t\} = \mathcal{O}(P)$, $E\{\underline{\alpha}_k\underline{\beta}_k^t\} = \mathcal{O}(P)$, $E\{\underline{\gamma}_k\underline{\alpha}_k^t\} = \mathcal{O}(P)$, $E\{\underline{\alpha}_k\underline{\gamma}_k^t\} = \mathcal{O}(P)$, $E\{\underline{\beta}_k\underline{\beta}_k^t\} = \mathcal{O}(P)$, $E\{\underline{\delta}_k\underline{\alpha}_k^t\} = \mathcal{O}(P^2)$, $E\{\underline{\alpha}_k\underline{\delta}_k^t\} = \mathcal{O}(P^2)$, $E\{\underline{\varepsilon}_k\underline{\alpha}_k^t\} = \mathcal{O}(P^2)$, $E\{\underline{\alpha}_k\underline{\varepsilon}_k^t\} = \mathcal{O}(P^2)$, $E\{\underline{\gamma}_k\underline{\beta}_k^t\} = \mathcal{O}(P^2)$, $E\{\underline{\beta}_k\underline{\gamma}_k^t\} = \mathcal{O}(P^2)$, $E\{\underline{\delta}_k\underline{\beta}_k^t\} = \mathcal{O}(P^2)$, $E\{\underline{\beta}_k\underline{\delta}_k^t\} = \mathcal{O}(P^2)$, $E\{\underline{\gamma}_k\underline{\gamma}_k^t\} = \mathcal{O}(P^2)$, while all other combinations yield orders in P higher than 2. Elaborating (12) and combining equal-order terms in P then yields a series expansion of V:

$$V = V_0 + V_1P + V_2P^2 + \cdots. \tag{13}$$

Notice that (in our normalized notation) for a vanishing input signal with $P \to 0$ a nonzero $V = V_0$ is found, clearly originating from the combination $E\{\underline{\alpha}_k\underline{\alpha}_k^t\}$.

For a given spectral and probability distribution the matrix coefficients V_0, V_1, V_2, $\cdots$ can be determined in a straightforward way, although the tedious analysis can yield rather complex results. Therefore we confine ourselves to some manageable special cases. For the most general distribution in frequency and amplitude we determine only V_0, while the combination $(V_0 + V_1P)$ is analyzed for x_k Gaussian and n_k white and, finally, $(V_0 + V_1P + V_2P^2)$ for x_k *and* n_k white and x_k Gaussian. Only the results are presented here, details will be published elsewhere [8].

3.1. The general case: x_k and n_k coloured

For the general excitation with arbitrarily coloured signals we determine V_0 as the solution of the Lyapounov equation

$$\begin{aligned} RV_0 + V_0R &= \sum_l N^{(l)}R^{(l)}, \\ N^{(l)} &= E\{n_kn_{k-l}\}, \; N^{(0)} = N, \\ R^{(l)} &= E\{\underline{x}_k\underline{x}_{k-l}^t\}, \; R^{(0)} = R. \end{aligned} \tag{14}$$

This recently published result [7] seems to have not yet received too much attention. For n_k white (14) admits a closed-form solution, viz. $V_0 = NI/2$, representing a set of equal-power, uncorrelated weight errors. In the more general case the Toeplitz structure of R and of the right-hand sum puts restrictions on V_0, among which a symmetry round the side diagonal [8], cf. Fig. 1. Further, the output signal y_k in the zero-order approximation can be substantially correlated with n_k. Contrary to common belief, the error signal e_k can even have a lower power than n_k, due to this correlation.

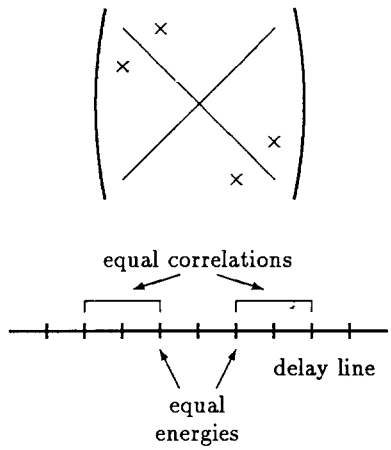

Figure 1: Symmetry in the correlation matrix V_0 (entries denoted by × are equal).

3.2. White reference signal n_k, Gaussian input signal x_k

Under these assumptions we find the combination

$$V_0 + V_1P = \frac{1}{2}N\{I(1 + \frac{1}{2}MP) + R\}. \tag{15}$$

In this first-order approximation, V has a Toeplitz structure with off-diagonal elements proportional to those of R. Then all pairs of neighbouring elements have the same correlation and the same result holds for all pairs with some given distance on the delay line (Fig. 2).

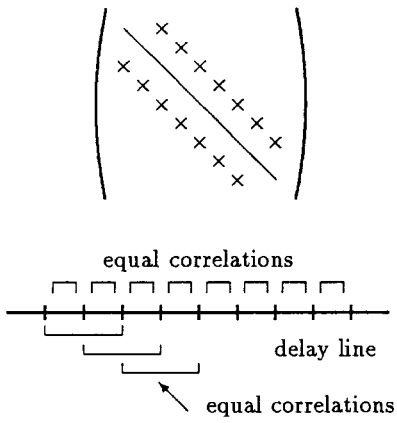

Figure 2: Symmetry in the correlation matrix $(V_0 + V_1 P)$ for a white reference signal.

3.3. White reference signal n_k, white and Gaussian input signal x_k

In this situation we obtain

$$V_0 + V_1 P + V_2 P^2 = \frac{N}{4}\left\{[2 + P(M+2) + \frac{P^2}{2}(M+2)^2]I - 2P^2 T - 4P^2 S\right\}. \quad (16)$$

$$T = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & & & & & \emptyset \\ & 1 & & & & \\ & & 2 & & & \\ & & & 3 & & \\ & & & & \ddots & \\ \emptyset & & & & & (M-1) \end{pmatrix}$$

Here we observe correlations between pairs of even-numbered weights and between pairs of odd-numbered weights, but not between mixed pairs. Further, the weights decay in amplitude along the delay line.

4. CONCLUSIONS

The weight-error correlation matrix has been determined as a power series in terms of the input power or (what amounts to the same) in terms of the adaptation constant. The strongest and most surprising effects are found if the input and the reference signal are coloured. Then in the zero-order approximation the WECM satisfies a Lyapounov equation yielding a set of weight fluctuations with diverse amplitudes and substantial mutual correlations. For a white reference signal the amplitudes become equal (even in the first-order approximation) and the mutual correlations are weak first-order effects. If also the input signal is white, the zero- and first-order theory predicts equal-power, uncorrelated weight fluctuations, while small correlations and a slight power decrease along the delay line are found in the second-order theory. All effects are confirmed by simulations [8] which have to be prepared with great care because otherwise the weak first- and second-order effects are obscured by statistical uncertainties.

5. REFERENCES

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