A NEW LOOK AT SIGNAL–ADAPTED QMF BANK DESIGN

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ABSTRACT

The design of a quadrature–mirror filter (QMF) bank (H, G) adapted to input signal statistics is considered. The adaptation criterion is maximization of the coding gain and has so far been viewed as a difficult nonlinear constrained optimization problem. In this paper, it is shown that in fact the coding gain depends only upon the product filter $P(z) = H(z)H(z^{-1})$. The optimization problem formulated in terms of the coefficients of $P(z)$ gives rise to a linear semi–infinite program (SIP). A simple SIP algorithm using a discretization method is presented. The filter $H(z)$ is obtained by deflation and spectral factorization of $P(z)$.

1. INTRODUCTION

Recently there has been growing interest in designing signal–adapted subband coding systems. Adaptivity may be obtained by designing filter banks that match the statistics of the input signal by maximizing the coding gain of the system [1, 2, 3, 4, 5, 6]. The output of the optimal filter bank enjoys an important decorrelation property [7]. As several experiments have demonstrated [2, 3], this technique has potential for higher compression efficiency than standard (nonadapted) subband/wavelet coding systems. Currently an obstacle to the use of such adaptive methods is the limited performance and substantial numerical complexity of the optimization algorithms involved.

We consider the two–band, perfect-reconstruction (PR), FIR, QMF bank displayed in Fig. 1. The lowpass and highpass filters $\{h_n\}$ and $\{g_n\}$ have length $2N$. The input signal x has length P and is replicated by periodic extension. The coding gain of the two-band system is given by $G_{TC} = \left(\frac{1}{2}(\sigma_h^2 + \sigma_g^2)\right)\left(\sigma_h^2\sigma_g^2\right)^{-1/2}$, where σ_h^2 and σ_g^2 are the variances of x_h and x_g. The maximimum of the coding gain is attained by maximizing $\mathcal{E}(h) \stackrel{\triangle}{=} \frac{1}{2}\sigma_h^2$ over the $2N$-vector h of coefficients $\{h_n\}$, subject to the PR constraints [2].

In the literature it is often assumed that the input process is stationary, and thus

$$\mathcal{E}(h) = \frac{1}{2} h^T \mathbf{R}\, h \tag{1}$$

where the $2N \times 2N$ covariance matrix $\mathbf{R}$ of the input signal is Toeplitz. Here the stationarity assumption will be replaced by a much weaker assumption: The time–averaged variances of the lowpass signal before and after subsampling are equal. Then

$$\mathcal{E}(h) = \frac{1}{2}\frac{1}{P}\sum_{p=0}^{P-1} E|x_h'(p)|^2 = \frac{1}{2}\frac{1}{P}\sum_{p=0}^{P-1} E\left|\sum_{n=0}^{2N-1} h_n\, x(p-n)\right|^2$$
$$= \frac{1}{2}\sum_{n,m=0}^{2N-1} h_n h_m E\left[\frac{1}{P}\sum_{p=0}^{P-1} x(p-n)x(p-m)\right].$$

Let $\mathbf{R}_{nm}$ be the expectation in the right–hand side. Since x is periodic, $\mathbf{R}_{nm}$ is a function of $|n-m|$ only. Thus, even though the input process x may be nonstationary, $\mathcal{E}(h)$ still takes the form (1) where $\mathbf{R}$ is Toeplitz, symmetric, and positive definite. The Toeplitz property is essential to the developments that follow.

2. MAXIMIZATION PROBLEM

The PR conditions take the form

$$\sum_{n=0}^{2N-1-2l} h_n h_{n+2l} = \delta_{l0}, \qquad 0 \le l < N \tag{2}$$

where δ_{mn} is the Kronecker delta. Maximization of (1) subject to the set of N quadratic constraints (2) is a difficult nonlinear optimization problem [4]. An alternative technique investigated in the literature is based on a lattice implementation of the QMF bank [8, 9]. The filter $\{h_n\}$ is then parameterized by a set of N angles $\{\alpha_k, 0 \le k < N\}$, and the optimization problem expressed in terms of $\{\alpha_k\}$ is unconstrained.

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Despite this natural parameterization, the existence of many local extrema and the nonlinearity of the equations present great difficulties.

Methods used in the literature for maximizing energy functions of the type (1) over $\{\alpha_k\}$ include quasi-Newton methods [8], the ring algorithm (a coordinate-descent scheme) [2], and multi-start algorithms [3]. The first two techniques result in convergence to a single local maximum that depends on the starting point of the iterative algorithm. The third technique yields a number of approximations to local maxima, by restricting the search to a discrete subset of the parameter space. In the next section, a more convenient formulation of the optimization problem is presented and a tractable algorithm converging to a global maximum is derived.

3. LINEAR SIP FORMULATION

Consider the product filter $P(z) \triangleq H(z)H(z^{-1})$. This filter is symmetric, has length $4N-1$ and, by application of (2), is parameterized by N coefficients $\{a_n, 0 \leq n < N\}$:

$$P(z) = 1 + \sum_{n=0}^{N-1} a_n(z^{-2n-1} + z^{2n+1}) \qquad (3)$$

where

$$a_n \triangleq \sum_{l=0}^{2N-2n-2} h_l h_{l+2n+1}. \qquad (4)$$

Since $P(f) + P(f+0.5) \equiv 2$, $P(f)$ is a half-band filter [10]. The necessary and sufficient conditions for $\{a_n\}$ to define a product filter are:

$$P(f) = |H(f)|^2 = 1 + 2\sum_{n=0}^{N-1} a_n \cos(2\pi f(2n+1)) \geq 0, \quad 0 \leq f \leq 0.5. \qquad (5)$$

Using the Toeplitz property of $\mathbf{R}$, we observe that the objective function in (1) is simply a linear function of $\{a_n\}$ in (4) and write it as

$$\mathcal{E}(a) = r_0/2 + \sum_{n=0}^{N-1} a_n r_{2n+1} \qquad (6)$$

where $\{r_n, 0 \leq n < 2N\}$ is the first row of $\mathbf{R}$. In other words, *the coding gain is a function of the product filter.*

Under the conditions (5), $H(z)$ may be obtained from $P(z)$ by spectral factorization. There are up to $2^N - 1$ different solutions corresponding to different groupings of zeroes of $P(z)$ [1] [11, p. 174]. The filter $H(z)$ can be designed to have minimum phase (all zeroes on or inside the unit circle) or close–to–linear phase (by suitable alternation of zeroes inside and outside the unit circle [10]). Phase design has no bearing on the coding gain.

By inspection of (5) and (6), the design problem formulated in terms of the product filter $P(z)$ is a linear optimization problem [12]. The coefficients $\{a_n\}$ belong to a closed, bounded, convex subset of $\mathcal{R}^N$ defined by the linear inequality constraints (5). The revisited optimization problem is more tractable than the original formulation (1)(2). In particular, every local solution is also a global solution. For certain values of $\mathbf{R}$, the solution may not be unique, e.g., for a white noise process all filters have identical performance: $\mathcal{E}(a) \equiv r_0/2$.

Because there is one linear constraint for each $f \in [0, 0.5]$, the linear programming problem is said to be of the semi–infinite type (SIP). Such problems occur in a variety of fields in applied science and economics [13]. A special type of linear SIP problem is Chebyshev approximation, which has well–known applications in filter design [14, 15] and may be solved using the powerful Remez exchange algorithm. However, the SIP problem (5)(6) is clearly not a Chebyshev approximation problem. Although fairly general SIP techniques may be applied, a specialized algorithm exploiting the special structure of (5)(6) may be simpler and more efficient. This is a topic of current research; meanwhile, we present a simple algorithm that was found to be reasonably efficient for low–to–moderate filter length (~ 20) and thus applicable to image coding.

4. ALGORITHM

For simplicity of this presentation, assume that the solution $\bar{a}$ to the SIP problem (5)(6) is unique (other cases are handled similarly). Denote by $\{\bar{f}_k, 0 \leq k < K\}$ the set of zeroes of the corresponding $P(f)$ in the interval $[0, 0.5]$. The number K is upper–bounded by $(N+1)/2$ [8]. An initial approximation to $\bar{a}$ is computed using a suitably defined discretization of the interval $[0, 0.5]$, and a feasible $\hat{a}$ for the SIP is subsequently computed. If the discretization of (5) is fine enough and certain regularity conditions are satisfied, $\hat{a}$ will be arbitrarily close to $\bar{a}$ [12, 16]. In practice however, the discretization should not be too fine otherwise numerical instabilities occur when $K < N$ [16].

The algorithm proceeds as follows.

Step 0. Define M frequencies $\{f_i = (2\pi)^{-1} \arccos(x_i), 0 \leq i \leq M\}$ where $x_i = -1 + 2i/M$ and M is an integer to be specified. This particular discretization allows the trigonometric functions $\cos(2\pi f_i$

[1] This explains the large number of local maxima of $\mathcal{E}$ as a function of h.

$(2n+1))$ to be efficiently evaluated by recursive computation of the Chebyshev polynomials $T_{2n+1}(x_i), 0 \leq n < N$ [14]. Define the discretized problem as (5)(6) where the inequality constraints are enforced only at $\{f_i\}$.

Step 1. Solve the dual program [12] $\min_\lambda \sum_{i=0}^{M} \lambda_i$ subject to

$$\begin{array}{rcll} -\sum_{i=0}^{M} \lambda_i \cos(2\pi f_i(2n+1)) & = & r_{2n+1}, & 0 \leq n < N, \\ \lambda_i & \geq & 0, & 0 \leq i \leq M. \end{array}$$

This is done using a double–precision version of the standard simplex algorithm in [17]. By the fundamental theorem of linear programming, λ has N nonzero components. For M large enough (and in the absence of numerical instability), these components are clustered around the zeroes $\{\bar{f}_k, 0 \leq k < K\}$. A typical example is illustrated in Fig. 2, with $N=4$ and $M=20$ ($K=2$).

Step 2. Compute a feasible $\hat{a}$ for the SIP problem (5)(6). This may be done as follows. Given $\{f_i | \lambda_i \neq 0\}$, the zeroes $\{\bar{f}_k, 0 \leq k < K\}$ and their (even) multiplicities $\{\mu_k, 0 \leq k < K\}$ are computed using a clustering technique. There is at most one k such that $\bar{f}_k \in \{0, 0.5\}$; if one such k exists, assume without loss of generality that $k = K-1$ and let $\mu_{K-1} \leftarrow \frac{\mu_{K-1}}{2}$. Assuming that the zeroes come in pairs but not in quadruples, etc..., the clustering technique simply consists in assigning $\bar{f}_k$ to the center of gravity of the pair (f_{2k}, f_{2k+1}). $P(z)$ has $2\sum_{k=0}^{K-1} \mu_k = 2N$ zeroes on the unit circle, including multiplicities. Finally the following system of N linear equations is solved for the N unknowns $\{a_n\}$:

$$\left\{ \begin{array}{ll} P^{(r)}(\bar{f}_k) = 0 & : \text{if } \bar{f}_k \in (0, 0.5) \\ P^{(2r)}(\bar{f}_k) = 0 & : \text{else} \end{array} \right.$$

$$0 \leq r < \mu_k, 0 \leq k < K.$$

Once $\{a_n\}$ is available, $H(z)$ is obtained by spectral factorization. Spectral factorization is an ill–conditioned problem when zeroes are close [11, p. 174][9]. However, some of these difficulties are alleviated by the fact that the zeroes of $P(z)$ on the unit circle are available. Thus, $P(z)$ may be deflated and spectral factorization performed on a polynomial of degree $2N-1$ only, instead of $4N-1$. We used Lang and Frenzel's spectral factorization software [18, 19], which provides an estimate for the relative accuracy of the zeroes in the complex plane. The accuracy was improved by up to several orders of magnitude using the deflation trick.

5. EXAMPLES

The algorithm was tested on three types of input signal:

1. AR(1) process with correlation coefficient $\rho = 0.95$ (simple image model [1, 5, 6]). In this case $r_n = \rho^n$.
2. AR(2) process with poles at $z_\pm = \rho e^{\pm i\theta}$, with $\rho = 0.975$ and $\theta = \pi/3$. Models certain types of image texture. Then $r_n = 2\rho\cos\theta\, r_{n-1} - \rho^2 r_{n-2}$, with $r_0 = 1$ and $r_1 = \frac{2\rho\cos\theta}{1+\rho^2}$.
3. lowpass process with spectral density $S(f) = rect_{[-f_s, f_s]}(f)$, with $f_s = 0.275$. The optimization criterion is then equivalent to minimization of the energy in the stopband $[f_s, 0.5]$ of $H(f)$ [8]. Then $r_n = \frac{\sin(2\pi f_s n)}{2\pi f_s n}$.

Results are shown in Tables 1 and 2 for $N=4$ and 10, respectively. As expected, the improvement over nonadapted filters is greatest when the signal statistics differ significantly from a lowpass model (e.g., AR(2) case) [3]. Even with double–precision arithmetic, numerical instabilities occur when M is too large (not shown here). When M is too small, the solution to the discretized problem may be unacceptably far from the exact solution, e.g., in the AR(1) example with $N=4$ and $M=20$, the "approximate" solution tends to infinity! The results obtained with $M \approx 20N$ were almost identical to those obtained with a more sophisticated SIP algorithm at University of Iowa, using Gribik's extension of the Elzinga-Moore central cutting plane method [13].

6. EXTENSIONS

The method has several extensions which will be described in more detail elsewhere.

1. *Forcing L zeroes of $H(f)$ at $f = 0.5$.* This implies L extra linear constraints on a,

   $$P^{(2r)}(0.5) = 0, \quad 0 \leq r < L.$$

   Then L variables, say $a_{N-L}, \cdots, a_{N-1}$, may be eliminated. The remaining variables are solution to a new $(N-L)$–dimensional SIP problem.
2. *Extension to M–band systems.*

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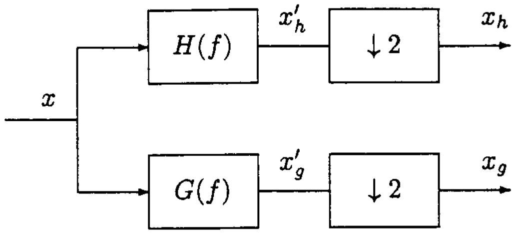

Figure 1: Two–channel analysis QMF bank.

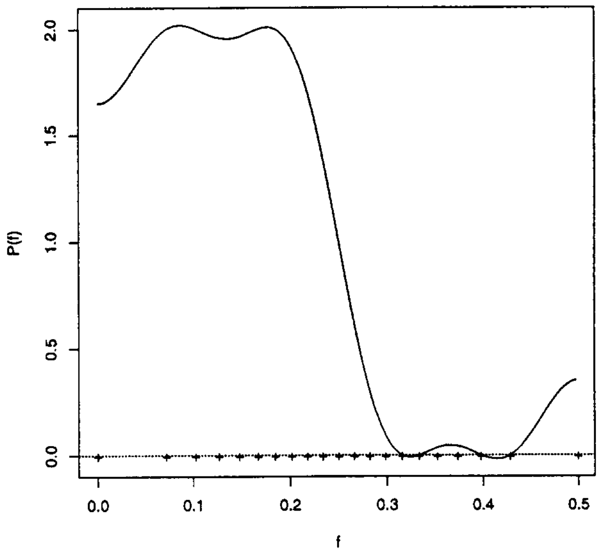

Figure 2: Frequency response of 8–tap filter ($N = 4$) for AR(2) example of Table 1, after Step 1 of the algorithm. Positivity constraints are enforced only at 21 frequencies as indicated by the symbols "+".

Process	Adapted filters			D8 [11]
	$M = 20$	$M = 50$	$M = 90$	
AR(1)	fails	5.846	5.859	5.810
AR(2)	5.642	6.068	6.070	2.632
lowpass	1.834	1.982	1.983	1.647

Table 1: Coding gain in dB for 8–tap filters.

Process	Adapted filters	
	$M = 90$	$M = 190$
AR(1)	2.790	5.943
AR(2)	6.827	6.835
lowpass	2.355	2.357

Table 2: Coding gain in dB for 20–tap filters.