DESIGN OF A CLASS OF MULTIRATE SYSTEMS

W. M. Campbell

Center for Applied Mathematics
Cornell University
Ithaca, NY 14853

T. W. Parks

Electrical Engineering
Cornell University
Ithaca, NY 14853

ABSTRACT

The design of rate-changing multirate systems using a maximum relative ℓ^2-error criterion is analyzed. Using multirate techniques, the criterion is simplified to a matrix-response approximation problem. An algorithm using convex optimization is proposed to solve the problem. An example illustrates the use of the algorithm and effectiveness over methods intended for LTI system design.

1. INTRODUCTION

Designing multirate systems such as the one shown in Figure 1 (commonly called rate-changing systems) with L and M relatively prime has been considered by many authors [1, 2]. Most approaches to the problem are motivated by either stochastic techniques, error with respect to a parameterized input class, or function approximation. Our technique attempts to minimize the maximum relative error with respect to a general class; this approach is useful if little is known about the signals to be processed. We note that our error measure has been proposed as a general criterion for multirate systems design in [3]; we analyze in detail the design of rate-changing multirate systems.

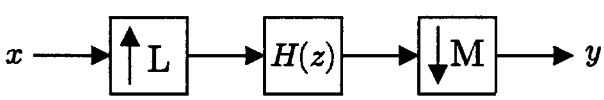

Figure 1: Multirate System to be Designed.

The criterion for the design of these systems is as follows. First, consider the error signal, w, produced by a FIR multirate system approximating an ideal multirate system for a given input, x, as shown in Figure 2.

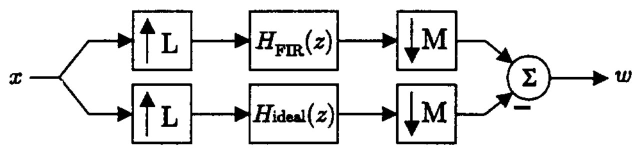

Figure 2: Comparison of Ideal and Approximate Multirate Systems.

A good performance measure for the FIR multirate system is the maximum relative error in norm:

$$\max_{x \neq 0} \frac{\|w\|_2}{\|x\|_2} \tag{1}$$

where for arbitrary x, $\|x\|_2 = \sqrt{\sum_{n=-\infty}^{\infty} |x(n)|^2}$ is the ℓ^2-norm. The design problem then is to find:

$$h_{\text{FIR,best}} = \underset{h_{\text{FIR}}}{\text{argmin}} \max_{x \neq 0} \frac{\|w\|_2}{\|x\|_2}. \tag{2}$$

This work was supported by the National Science Foundation, Grant MIP9224424 and ONR, Grant N00014-1-0102.

This criterion is a natural generalization of the standard Chebyshev error [3] and is commonly used in H^∞-control theory [4, 5].

2. ANALYSIS

2.1. Simplification of the Error Criterion

In this section, we put (1) in a more manageable form. The system in Figure 1 is equivalent to Figure 3, [6].

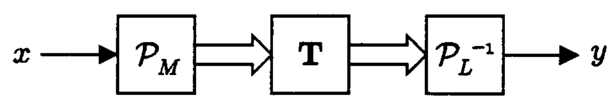

Figure 3: Matrix Form of the Multirate System.

$\mathbf{T}(z) = [T_{i,j}(z)]$ is an $L \times M$ matrix of filters,

$$t_{i,j}(n) = h(MLn + Mi - Lj) \tag{3}$$

$i = 0, \ldots, L-1$, $j = 0, \ldots, M-1$. $\mathcal{P}_M$ is the M-polyphase decomposition and $\mathcal{P}_L^{-1}$ is its inverse; i.e., if

$$X(z) = \sum_{i=0}^{M-1} z^{-i} X_i(z^M), \tag{4}$$

then

$$[\mathcal{Z}(\mathcal{P}_M(x))](z) = \begin{bmatrix} X_0(z) & \ldots & X_{M-1}(z) \end{bmatrix}^T. \tag{5}$$

$\mathcal{Z}$ indicates the z-transform. The double arrow in Figure 3 indicates a vector signal.

Because $\mathcal{P}_{(\cdot)}$ is unitary, the error criterion (1) can be expressed as

$$\max_{0 \leq f \leq 1} \|\mathbf{T}_{\text{FIR}}(f) - \mathbf{T}_{\text{ideal}}(f)\|_2 \tag{6}$$

0-7803-2431-5/95 $4.00 © 1995 IEEE

where 'max' is used since the normed matrix function is assumed to be piecewise continuous with the maximum occurring on one of the pieces.

We now relate the matrix form of Figure 3 to the standard representation in Figure 1. The ith L-polyphase component of the output, y_i, is obtained from the ith row of the matrix as shown in Figure 4(a). This structure simplifies to Figure 4(b).

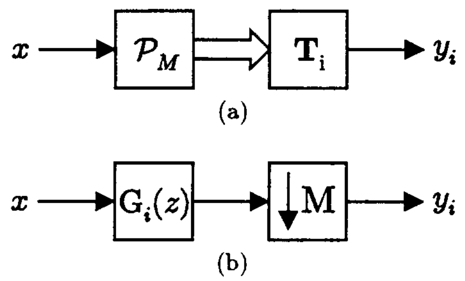

Figure 4: (a) One row of the matrix. (b) Rearrangement of the row.

Using Figure 4, Figure 3 simplifies to Figure 5 where $\mathbf{G} = \begin{bmatrix} G_0 & \dots & G_{L-1} \end{bmatrix}^T$ and

$$G_{i,j}(z) = \begin{cases} T_{i,0}(z) & j = 0 \\ zT_{i,M-j}(z) & j = 1, \dots, M-1. \end{cases} \tag{7}$$

We refer to Figure 5 as the ***canonical form***. Here $G_{i,j}$ is the jth M-polyphase component of G_i. The G_i are related to H in Figure 1 (see [7]) by

$$G_i(z) = z^{q_i} H_{r_i}(z) \tag{8}$$

where $Mi = q_i L + r_i$, $0 \le r_i < L$, $0 \le i < L-1$, and H_{r_i} is the r_ith L-polyphase component of H.

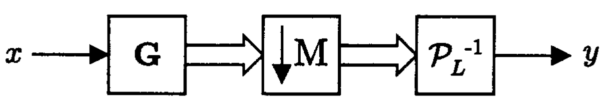

Figure 5: Canonical form of the sampling-rate converter.

The criterion (6) can now be related to H_{ideal} and H_{FIR} using the modulation representation [8] (or alias-component form) and equations (7) and (8). For the filters given by (8) let $\mathbf{G}_{\text{mod}}$ be

$$\begin{bmatrix} G_0(f) & G_0(f+\frac{1}{M}) & \dots & G_0(f+\frac{M-1}{M}) \\ \vdots & \dots & \dots & \vdots \\ G_{L-1}(f) & G_{L-1}(f+\frac{1}{M}) & \dots & G_{L-1}(f+\frac{M-1}{M}) \end{bmatrix}. \tag{9}$$

Now let $\mathbf{G}_{\text{mod}}^{\text{FIR}}$ and $\mathbf{G}_{\text{mod}}^{\text{ideal}}$ be the matrices given by (9) for H_{FIR} and H_{ideal}, respectively, in Figure 2. Then (6) (and thus (1)) can be expressed as

$$\max_{0 \le f \le \frac{1}{M}} \frac{1}{\sqrt{M}} \|\mathbf{G}_{\text{mod}}^{\text{FIR}}(f) - \mathbf{G}_{\text{mod}}^{\text{ideal}}(f)\|_2. \tag{10}$$

Now write each G_i^{FIR} as

$$G_i^{\text{FIR}}(f) = \sum_{k=-N_i}^{M_i} g_{i,k} e^{-j2\pi k f}. \tag{11}$$

Define $\mathbf{E}_{i,k}$ to be the matrix whose (i,k)-th entry is 1 and has zeros elsewhere. Also, define

$$\mathbf{X}_{i,k}(f) = \sum_{l=0}^{M-1} e^{-j2\pi k(f+\frac{l}{M})} \mathbf{E}_{i,l}. \tag{12}$$

Then

$$\mathbf{G}_{\text{mod}}^{\text{FIR}}(f) = \sum_{i=0}^{L-1} \sum_{k=-N_i}^{M_i} g_{i,k} \mathbf{X}_{i,k}(f). \tag{13}$$

Define

$$\mathbf{g}' = \begin{bmatrix} g_{0,-N_0} \\ \vdots \\ g_{0,M_0} \\ \vdots \\ g_{L-1,-N_{L-1}} \\ \vdots \\ g_{L-1,M_{L-1}} \end{bmatrix} \tag{14}$$

$$\mathbf{g} = \begin{bmatrix} \text{Re}(\mathbf{g}') \\ \text{Im}(\mathbf{g}') \end{bmatrix} \tag{15}$$

$$e(\mathbf{g}) = \frac{1}{\sqrt{M}} \max_{0 \le f \le \frac{1}{M}} \|\sum_{i=0}^{L-1} \sum_{k=-N_i}^{M_i} g_{i,k} \mathbf{X}_{i,k}(f) - \mathbf{G}_{\text{mod}}^{\text{ideal}}(f)\|_2. \tag{16}$$

Then the design problem (2) becomes

$$\mathbf{g}_{\text{best}} = \operatorname*{argmin}_{\mathbf{g}} e(\mathbf{g}). \tag{17}$$

2.2. Analysis of the Error Functional

The error functional, e, given by (16) is convex. This crucial fact is the basis for our solution method detailed in the next section. In order to use this method, the subdifferential [9] of $e(\mathbf{g})$ is needed. We state some definitions and then give the subdifferential.

Define an inner product between two $L \times M$ matrices $\mathbf{C}$ and $\mathbf{D}$ by

$$\langle \mathbf{C}, \mathbf{D} \rangle = \text{tr}(\mathbf{D}^H \mathbf{C}) = \sum_{i=0}^{L-1} \sum_{j=0}^{M-1} c_{i,j} d_{i,j}^*. \tag{18}$$

Define

$$\mathbf{X}_k(f) = \begin{bmatrix} e^{-j2\pi k(f+\frac{0}{M})} \\ \vdots \\ e^{-j2\pi k(f+\frac{M-1}{M})} \end{bmatrix}. \tag{19}$$

Then

$$\partial e(\mathbf{g}) = \text{co}\{\mathbf{a} | a_{i,k} = \frac{1}{\sqrt{M}} \mathbf{X}_k(f')^H \delta'_i,\ (\delta', f') \in \mathcal{S}(\mathbf{g})\} \tag{20}$$

where

$$\mathcal{S}(\mathbf{g}) = \{(\delta', f') | \operatorname{Re}(\langle \mathbf{G}^{\text{FIR}}_{\text{mod}}(\mathbf{g}, f') - \mathbf{G}^{\text{ideal}}_{\text{mod}}(f'), \delta' \rangle) = e(\mathbf{g})\}. \tag{21}$$

In (20), "co" indicates the convex hull, and δ'_i denotes the transpose of the ith row of δ'. Explicit dependence of $\mathbf{G}^{\text{FIR}}_{\text{mod}}$ on $\mathbf{g}$ is also indicated. Finally, the indexing for $\mathbf{a}$ is specified in the same way as for $\mathbf{g}$ in equations (14) and (15); if a *real* FIR solution is desired, then the $g_{i,k}$'s are real, and the "Im" part should be omitted in $\mathbf{g}$ and $\mathbf{a}$.

3. SOLUTION METHOD

Our solution method is based on recent advances in convex optimization. We have implemented the algorithm described in [10] for convex optimization. This method is part of a general class of methods known as ***bundle methods*** [11]. These methods optimize by bundling information about the function and its subdifferential to obtain descent directions. The method dynamically constructs a piecewise linear approximation, $f_{cp,k}$, of the function to be optimized. A local optimization involving $f_{cp,k}$ is then solved using trust region methods. This optimization produces a direction which is either used to gather more local information or as a descent step. For more details, we refer to [10].

We now describe our methods of calculation for the data needed in the algorithm. The first datum that needs to be calculated is $e(\mathbf{g})$. For fixed $\mathbf{g}$ define

$$N(f) = \frac{1}{\sqrt{M}} \|\mathbf{G}^{\text{FIR}}_{\text{mod}}(\mathbf{g}, f) - \mathbf{G}^{\text{ideal}}_{\text{mod}}(f)\|_2 \tag{22}$$

Dependence of N on $\mathbf{g}$ will be indicated by writing $N(\mathbf{g}, f)$. We calculate on a grid of frequencies $\{f_i\}$ the maximum of $N(\mathbf{g}, f_i)$; this gives a suitable approximation to $e(\mathbf{g})$. Usually the response of the ideal system is given in term of the filter H_{ideal}. By calculating $H_{\text{ideal}}(f)$ on a grid and using (8) and the modulation representation, one can find the responses $\{G_i^{\text{ideal}}\}$ of the canonical form filters.

The other datum to be calculated is a subgradient at a given point, $\mathbf{g}$. Suppose $N(\mathbf{g}, f)$ obtains its maximum at f'. Then calculate any normalized left singular vector, $\mathbf{u}$, and its corresponding normalized right singular vector, $\mathbf{v}$, of $\frac{1}{\sqrt{M}} \mathbf{G}^{\text{FIR}}_{\text{mod}}(\mathbf{g}, f') - \frac{1}{\sqrt{M}} \mathbf{G}^{\text{ideal}}_{\text{mod}}(f')$ which belongs to the singular value $N(\mathbf{g}, f')$. The formula for $\mathbf{a}$ which appears in equation (20) with $\delta' = \mathbf{u}\mathbf{v}^H$ then gives a subgradient in $\partial e(\mathbf{g})$.

Finally, we comment on several aspects of our method. First, the method is globally convergent; this property is shown in [10]. Second, an acceptable convergence rate is observed in practice; this feature is mentioned in [10] and was also observed for the optimizations performed. Third, local convergence might be accelerated using optimization methods tailored to the structure of the problem. Many advances have been made towards this goal [12, 13].

4. EXAMPLE

We now consider the design of an optimal H_{FIR} for the system in Figure 2 with $L = 2$ and $M = 5$. Let H_{FIR} be linear phase, length 101, and have real coefficients. Let H_{ideal} be linear phase and have group delay 50. Figure 6(a) shows the magnitude of the desired response of $H_{\text{ideal}}(f)$.

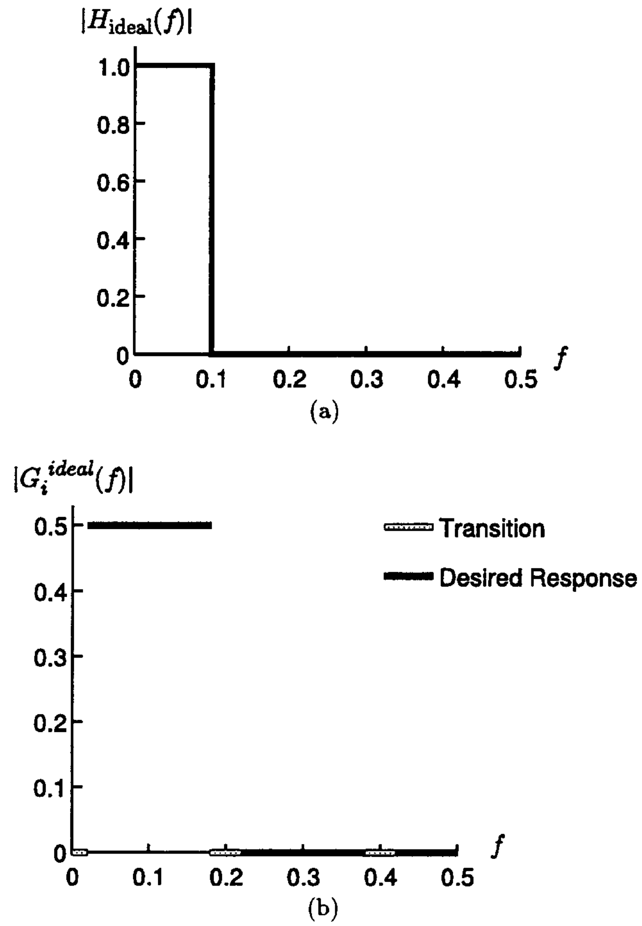

Figure 6: (a) Magnitude of the Frequency Response of the Ideal Filter. (b) Magnitude of the Frequency Responses of the Canonical Form Filters.

We specify the transition region for the ideal filters $\{G_i^{\text{ideal}}\}$ since they are directly related to the matrix form. First, we note that $|G_i^{\text{ideal}}(f)|$ is the same for all i. Suppose a transition region of $[.18, .20]$ is desired for $|G_i^{\text{ideal}}(f)|$. Then because of the periodicity of the entries in (9) and the conjugate symmetry of the responses (because the filters are real-coefficient), the transition region must be enlarged to $[0, .02] \cup [.18, .22] \cup [.38, .42]$. The resulting response is shown in Figure 6(b).

The magnitude of the frequency response of the resulting optimal H_{FIR} is shown in Figure 7. Figure 8 shows $N(f)$ for the optimal H_{FIR} in $[.02, 0.18]$ (i.e., $[0, \frac{1}{M}]$ minus the transition region). The maximum relative error is 0.00441. For comparison, a length 101 linear-phase filter was designed with the Parks-McClellan algorithm [14] using the transition region $[0,.01] \cup [.09,.11] \cup [.19,.21] \cup [.29,.31] \cup [.39,.41] \cup [.49,.5]$; the maximum relative error in this case is 0.00725. The dotted line in Figure 8 shows $N(f)$ for the Chebyshev design. Note that the error is decreased by 64 percent with this new design.

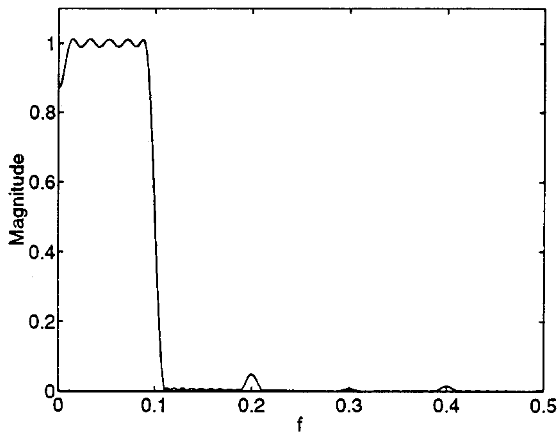

Figure 7: Magnitude of the Frequency Response of the Optimal H_{FIR}.

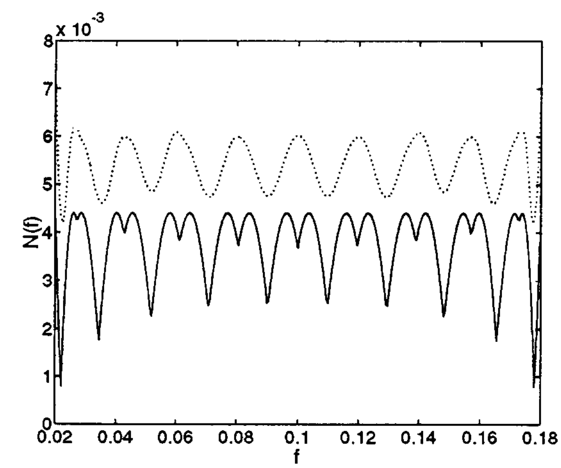

Figure 8: Solid Line—Plot of $N(f)$ for the Optimal System. Dotted Line—Plot of $N(f)$ for a Chebyshev Design.

5. SUMMARY

We have presented new analysis and a new design method for multirate systems for rate-changing. The analysis involved the rearrangement of the design problem to a matrix response approximation problem and an analysis of the resulting error functional. The design method involved the formulation and solution of the approximation problem as a convex optimization problem. An example illustrated many aspects of the new method including gains over traditional Chebyshev-based methods. The application of these methods to other multirate systems is currently being investigated.

6. REFERENCES

[1] P. P. Vaidyanathan, ***Multirate Systems and Filter Banks.*** Prentice-Hall, Englewood Cliffs, NJ, 1993.

[2] R. E. Crochiere and L. R. Rabiner, ***Multirate Digital Signal Processing.*** Prentice-Hall, Englewood Cliffs, NJ, 1983.

[3] R. G. Shenoy, D. Burnside, and T. W. Parks, "Linear periodic systems and multirate filter design," ***IEEE Trans. Signal Processing***, vol. 42, pp. 2242–2256, Sept. 1994.

[4] B. A. Francis, ***A Course in H_∞ Control Theory.*** Springer-Verlag, 1987.

[5] T. Chen and L. Qiu, "H_∞ design of general multirate sampled-data control systems," ***Automatica***, vol. 30, no. 7, pp. 1139–1152, 1994.

[6] T. Miyawaki and C. W. Barnes, "Multirate recursive digital filters—a general approach and block structures," ***IEEE Trans. Acoust., Speech, Signal Processing***, vol. ASSP-31, pp. 1148–1154, Oct. 1983.

[7] J. Kovačević and M. Vetterli, "Perfect reconstruction filter banks with rational sampling factors," ***IEEE Trans. Signal Processing***, vol. 41, pp. 2047–2066, June 1993.

[8] M. Vetterli, "A theory of multirate filter banks," ***IEEE Trans. Acoust., Speech, Signal Processing***, vol. ASSP-35, pp. 356–372, Mar. 1987.

[9] R. T. Rockafellar, ***Convex Analysis.*** Princeton University Press, 1970.

[10] H. Schramm and J. Zowe, "A version of the bundle idea for minimizing a nonsmooth function," ***SIAM J. Optimization***, vol. 2, pp. 121–152, Feb. 1992.

[11] J.-B. Hiriart-Urruty and C. Lemaréchal, ***Convex Analysis and Minimization Algorithms, vols. I and II.*** Springer-Verlag, 1993.

[12] M. L. Overton, "On minimizing the maximum eigenvalue of a symmetric matrix," ***SIAM J. Matrix Anal. Appl.***, vol. 9, pp. 256–268, Apr. 1988.

[13] M. K. H. Fan, "A quadratically convergent local algorithm on minimizing the largest eigenvalue of a symmetric matrix," ***Linear Algebra and Its Applications***, vol. 188, 189, pp. 231–253, 1993.

[14] T. W. Parks and C. S. Burrus, ***Digital Filter Design.*** Wiley-Interscience, 1987.